proprep

Workbook

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The Argument Principle

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Questions

1) Compute $\int\limits_{|z|=2} \frac{2z}{z^2+1}dz$ using the Argument Principle.

Reminder: $\oint\limits_{\gamma} \frac{f'(z)}{f(z)}dz = 2\pi i(N-P)$.

2) Let $f(z)=\frac{(z+3)(z+1)}{z^2}$. Compute $\int\limits_{|z|=2} \frac{f'(z)}{f(z)}dz$ using the Argument Principle.

Reminder: $\oint\limits_{\gamma} \frac{f'(z)}{f(z)}dz = 2\pi i(N-P)$

3) Compute $\int\limits_{|z|=1} \frac{3\sin 2z}{\sin^2 z-0.5}dz$ using the Argument Principle.

Reminder: $\oint\limits_{\gamma} \frac{f'(z)}{f(z)}dz = 2\pi i(N-P)$

4) Let $f(z)=\frac{(z-2)^2}{\left(e^{2z}-1\right)^2 z^3}$.

a. Find the zeros and poles of f (and their orders) in the domain $|z|<4$.

b. Compute $\frac{1}{2\pi i}\int\limits_{|z|=4} \frac{f'(z)}{f(z)}dz$ using the Argument Principle.

Reminder: $\frac{1}{2\pi i}\oint\limits_{\gamma} \frac{f'(z)}{f(z)}dz = N-P$.

Answer Key

1) $4\pi i$
2) $-2\pi i$
3) $12\pi i$
4) a. 0 is a pole order 5 and $\pm\pi i$ are poles of order 2 b. -7

The Argument Principle - Geometric Meaning

Questions

1) Let $f(z)=z^2$ and $\gamma=\{z=e^{i\theta}:0\le\theta\le2\pi\}=\{z:|z|=1\}$.

Show that the equality $\frac{1}{2\pi i}\oint_\gamma \frac{f'(z)}{f(z)}dz=\frac{1}{2\pi}\Delta\arg_\gamma f(z)$ holds,

by evaluation each side separately and comparing.

2) Compute $\frac{1}{2\pi}\Delta\arg_\gamma f(z)$, where $f(z)=z^2-z$ and $\gamma=\{z=2e^{i\theta}:0\le\theta\le2\pi\}$.

3) Let $f(z)=e^z$ and $\gamma=\{z=e^{i\theta}:0\le\theta\le2\pi\}=\{z:|z|=1\}$.

Show that the equality $\frac{1}{2\pi i}\oint_\gamma \frac{f'(z)}{f(z)}dz=\frac{1}{2\pi}\Delta\arg_\gamma f(z)$ holds,

by evaluation each side separately and comparing.

4) Use the Argument Principle to compute N for the polynomial $p(z)=z^2+1$ in $\operatorname{Im} z>0$ (the upper half-plane).

5) Use the Argument Principle to compute the number of zeros N for the polynomial $p(z)=z^4+8z^3+3z^2+8z+3$ in $\operatorname{Re} z>0$ (the right half-plane).

Answer Key

1) $\frac{1}{2\pi i}\oint_{\gamma}\frac{f'(z)}{f(z)}dz=2$, $\frac{1}{2\pi}\Delta\arg_{\gamma} f(z)=2$
2) $\frac{1}{2\pi i}\oint_{\gamma}\frac{f'(z)}{f(z)}dz=N-P=2-0=2$
3) $\frac{1}{2\pi i}\oint_{\gamma}\frac{f'(z)}{f(z)}dz=0$, $\frac{1}{2\pi}\Delta\arg_{\gamma} f(z)=0$
4) $N=1$
5) $N=2$

Rouche's Theorem

Questions

1) Find the number of zeros of $h(z)=z^4+5z+1$ in $D=\{z\in\mathbb{C}:|z|<1\}=D(0,1)$.

2) Find the number of zeros of $h(z)=5z^4+z+1$ in $D=\{z\in\mathbb{C}:|z|<1\}=D(0,1)$.

3) Find the number of zeros of $h(z)=z^4+5z+4$ in $D=\{z\in\mathbb{C}:|z|<2\}=D(0,2)$.

4) Find the number of zeros of $h(z)=z^5+3z+1$ in $D=\{z\in\mathbb{C}:1<|z|<2\}$.

5) Find the number of zeros of $h(z)=z^4-10z+1$ in $D=\{z\in\mathbb{C}:1<|z|<3\}$.

6) Find the number of zeros of $h(z)=\frac{1}{2}z^6-5z^4+7z$ in $D=\{z\in\mathbb{C}:|z|<1\}=D(0,1)$.

7) Find the number of zeros of $h(z)=\frac{1}{2}z^6-5z^4+7z$ in $D=\{z\in\mathbb{C}:1<|z|<3\}$.

8) Let $m\geq 2$ Find the number of zeros (including multiplicity) in $D=\{z\in\mathbb{C}:|z|<1\}=D(0,1)$ of $h(z)=ze^{m-z}-1$.

9) Let $D=\{z\in\mathbb{C}:|z|<1\}=D(0,1)$ and let $a_1,\dots,a_n\in D$ be all different.

Denote $B(z)=\prod_{j=1}^{n}\frac{z-a_j}{1-z\overline{a_j}}=\frac{z-a_1}{1-z\overline{a_1}}\cdot\ldots\cdot\frac{z-a_n}{1-z\overline{a_n}}$ and let $z_0\in D$.

Prove that the function $B(z)-z_0$ has n roots in D (including multiplicities).

Hint: show that $\varphi_j(z)=\frac{z-a_j}{1-z\overline{a_j}}$ maps the unit circle into itself.

10) Prove that there exists a number $N\in\mathbb{N}$ such that, for all $n>N$ $(n\in\mathbb{N})$, there is exactly one solution of the equation $\sin z=z^n$ in the disk $D=D(0,\frac{1}{2})$.

11) Let f be analytic on the closed unit disk $\overline{D}(0,1)$ and satisfy $|f(z)|<1$ for all $|z|=1$. Prove that f has exactly one stationary point in the open unit disk. In other words, prove that the equation $f(z)=z$ has exactly one solution in $D(0,1)$.

Answer Key

1) 1
2) 4
3) 4
4) 4
5) 3
6) 1
7) 3
8) 1
9) Proof
10) Proof
11) Proof

Hurwitz s Theorem

Questions

12) Prove that there exists $N \in \mathbb{N}$ such that, for all $n > N$, the equation $z - \frac{z^3}{3!} + \frac{z^5}{5!} - ... + (-1)^{n+1}\frac{z^{2n+1}}{(2n+1)!} = 0$ has exactly one solution in $D = \{z : |z - \pi| < 1\}$.

Hint: Use Hurwitz's theorem and the Taylor series for $\sin z$.

13) Let $R > 0$. Prove that there exists $N \in \mathbb{N}$ such that, for all $n > N$, the equation $1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + ... + \frac{z^n}{n!} = 0$ has no solutions in $D = D(0, R) = \{z : |z| < R\}$.

Hint: Use Hurwitz's theorem and the Taylor series for e^z.

*Proof questions- for the full solution see the video

Existence of Logarithms and Roots of Functions

Questions

1) Show that the function $f(z)=\frac{z-1}{z+1}$ has an analytic logarithm in $D=\mathbb{C}-\{(-\infty,-1]\cup[1,\infty)\}$.

2) Let D be the domain $\mathbb{C}-[-i,i]$, as illustrated.

a. Prove that the function $f(z)=\frac{z-i}{z+i}$ has an analytic logarithm on D.

b. Prove that the function $F(z)=z^2\frac{z-i}{z+i}$ has an analytic square root on D.

3) Let D be the domain $\mathbb{C}-[-1,1]$, and let $f(z)=1-z^2$ on D.

a. Prove that $f(z)$ does **not** have an analytic logarithm on D.

b. Prove that $f(z)$ **does** have an analytic square root on D.

4) Let D be the domain $\{z:|z|>4\}$, and let $f(z)=(z-2)^2-4$ on D.

a. Prove that $f(z)$ does **not** have an analytic logarithm on D.

b. Prove that $f(z)$ **does** have an analytic square root on D.

*Proof questions- for the full solution see the video