

Workbook



Table of Contents

Schwarz's Lemma	2
Schwarz's Lemma.....	2

Schwarz's Lemma

Schwarz's Lemma

Questions

- 1) Let $f, g : D(0,1) \rightarrow D(0,1)$ be holomorphic equivalences and suppose that, for some $z_1 \in D(0,1)$, $f(z_1) = 0$ and $g(0) = z_1$. Reminder: $D(0,1) = \{z \in \mathbb{C} : |z| < 1\}$. Show that $|f(z)| \leq |g^{-1}(z)|$ for all $z \in D(0,1)$.
- 2) Let $f, g : D(0,1) \rightarrow D(0,1)$ be holomorphic equivalences and suppose that, for some $z_1 \in D(0,1)$, $f(z_1) = g(z_1) = 0$.
- Show that $|f(z)| \leq |g(z)|$ for all $z \in D(0,1)$.
 - Deduce from the above that $|f(z)| = |g(z)|$ for all $z \in D(0,1)$.
- 3) Let g be the Möbius Transformation $g(z) = \frac{z-a}{1-\bar{a}z}$ where $a \in D(0,1)$.
- Show that if $|z| = 1$ then $|g(z)| = 1$; i.e., g maps the unit circle to itself.
 - Deduce from the above that g maps $D(0,1)$ to itself: $g : D(0,1) \rightarrow D(0,1)$.
 - Let $f : D(0,1) \rightarrow D(0,1)$ be a holomorphic equivalence which satisfies $f(a) = 0$
- Prove that, for all $z \in D(0,1)$, $|f(z)| = \left| \frac{z-a}{1-\bar{a}z} \right|$. [Hint: use a previous exercise.]
- 4) Let g be the Möbius Transformation $g(z) = \frac{z-a}{z-\bar{a}}$, where $a \in H^+ = \{z \in \mathbb{C} \mid \text{Im } z > 0\}$.
- Show that g maps the real axis to the unit circle $C = \{z \in \mathbb{C} : |z| = 1\}$.
 - Deduce from the above that g maps H^+ to $D(0,1)$.
 - Let $f : H^+ \rightarrow D(0,1)$ be a holomorphic equivalence which satisfies $f(a) = 0$.
- Prove that $|f(z)| \leq \left| \frac{z-a}{z-\bar{a}} \right|$ for all $z \in H^+$.

Schwarz's Lemma

- 5) Prove the following generalization of Schwarz's Lemma. Suppose that:
- $f(z)$ is holomorphic in $D(0,1)$.
 - For all $z \in D(0,1)$, $|f(z)| \leq 1$
 - $f(0) = f^{(1)}(0) = \dots = f^{(m-1)}(0) = 0$ for some natural number $m \geq 1$.

Then:

- For all $z \in D(0,1)$, $|f(z)| \leq |z|^m$.
- 6) Denote $D = \{z \in \mathbb{C} : |z| < 1\}$ and $H^r = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$.
- Define the Möbius Transformation $\varphi(z) = \frac{z - \alpha}{z + \bar{\alpha}}$, where $\alpha \in H^r$, arbitrary.
Show that $\varphi(H^r) = D$.
 - Let $f : H^r \rightarrow D$ be analytic and suppose that $f(\alpha) = f'(\alpha) = 0$ for some $\alpha \in H^r$.
Prove that $|f(w)| \leq \left| \frac{w - \alpha}{w + \bar{\alpha}} \right|^2$ for all $w \in H^r$.
 - Let $f : H^r \rightarrow D$ be analytic and suppose $f(\alpha) = f(\beta) = 0$, where $\alpha, \beta \in H^r$ and $\alpha \neq \beta$.
Prove that $|f(w)| \leq \left| \frac{w - \alpha}{w + \bar{\alpha}} \right| \left| \frac{w - \beta}{w + \bar{\beta}} \right|$ for all $w \in H^r$.

- 7) Denote $D = \{z \in \mathbb{C} : |z| < 1\}$. Let $f : D \rightarrow D$ be analytic.

- Prove that for all $z_1, z_2 \in D$, $\left| \frac{f(z_2) - f(z_1)}{1 - \overline{f(z_2)}f(z_1)} \right| \leq \left| \frac{z_2 - z_1}{1 - \overline{z_2}z_1} \right|$
- Prove that for all $z \in D$, $\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}$

Hint: use the mapping $\varphi_a : D \rightarrow D$, $\varphi_a(z) = \frac{z - a}{1 - \bar{a}z}$.

- 8) Denote $D = \{w \in \mathbb{C} : |w| < 1\}$ and $H^+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$.

Let $f : H^+ \rightarrow H^+$ be analytic and let $z_0 \in H^+$.

Prove that $\left| \frac{f(z) - f(z_0)}{f(z) - \overline{f(z_0)}} \right| \leq \left| \frac{z - z_0}{z - \bar{z}_0} \right|$, $\forall z \in H^+$.

Hint: use the mapping $\varphi_a : H^+ \rightarrow D$, $\varphi_a(z) = \frac{z - a}{z - \bar{a}}$, $a \in H^+$.

Schwarz's Lemma

9) Let $f : D(0,1) \rightarrow D(0,1)$ be analytic and let $\alpha, \beta \in D(0,1)$ be two different fixed points of f . In other words, $\alpha \neq \beta$, $f(\alpha) = \alpha$, $f(\beta) = \beta$. Prove that $f(z) \equiv z$.

10) Let $f : D(0,1) \rightarrow D(0,1)$ be analytic. Prove that $|f'(z)| \leq \frac{1}{1-|z|^2}$ for all $z \in D(0,1)$.

*Proof questions- for the full solution sww the video