proprep

# Workbook

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# Complex Number

## Introduction to complex Numbers

## Questions

**1)** Express in terms of $i$:

a. $\sqrt{-1} =$ b. $\sqrt{-4} =$ c. $\sqrt{-25} =$

d. $\sqrt{-3} =$ e. $\sqrt{-5} =$

**2)** Compute:

a. $i^1$ b. $i^2$ c. $i^3$

d. $i^4$ e. $i^5$ f. $i^{17}$

**3)** Write the values of $a$ and $b$ for the following complex numbers:

a. $2+5i$ b. $3-i$ c. $\frac{\sqrt{3}}{2}-\frac{1}{2}i$

d. $7i$ e. $-4$ f. $0$

**4)** Solve the following quadratic equations:

a. $x^2=-1$ b. $x^2+36=0$ c. $x^2-2x+5=0$

**5)** Solve the following quadratic equation: $x^2+x+1=0$.

**6)** Solve the following quadratic equation: $z^2+iz+6=0$.

**7)** Given two complex numbers: $z_1 = 2+3i$ , $z_2 = 5-2i$ .
Compute the value of the following expressions:

a. $z_1 + z_2$ b. $z_1 - z_2$ c. $z_1 \cdot z_2$

**8)** Write the complex conjugate of the following complex numbers:

a. $2+5i$ b. $3-i$ c. $\frac{\sqrt{3}}{2} - \frac{1}{2}i$

d. $7i$ e. $-4$ f. $0$

**9)** Compute:

a. $\frac{11+2i}{2-i}$ b. $\frac{3+7i}{2-5i}$ c. $\frac{19-9i}{2-3i}$

## Answer Key

**1)** a. $i$ b. $2i$ c. $5i$ d. $\sqrt{3i}$ e. $\sqrt{5i}$

**2)** a. $i$ b. $-1$ c. $-i$ d. $1$ e. $i$ f. $i$

**3)** a. $a=2,\ b=5$ b. $a=3,\ b=-1$ c. $a=\frac{\sqrt{3}}{2},\ b=-\frac{1}{2}$

d. $a=0,\ b=7$ e. $a=-4,\ b=0$ f. $a=0,\ b=0$

**4)** a. $x=\pm i$ b. $x=\pm 6i$ c. $x=1\pm 2i$

**5)** $x=z=-\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$

**6)** $z_1=2i,\ z_2=-3i$

**7)** a. $7+i$ b. $-3+5i$ c. $16+11i$

**8)** a. $2-5i$ b. $3+i$ c. $\frac{\sqrt{3}}{2}+\frac{1}{2}i$ d. $-7i$ e. $-4$ f. $0$

**9)** a. $4+3i$ b. $-1+i$ c. $5+3i$

## Complex Equations and solutions

## Questions

**1)** Solve the following equation: $3z-11=iz-7i$.

**2)** Solve the following equation: $iz+5=4i$.

**3)** Solve the following system of equations in two complex unknowns, $z$ and $w$: $\begin{cases}3z+iw=5-4i\\5iz-2w=5+8i\end{cases}$.

**4)** Solve the following equations, in which $a$ and $b$ are real:

a. $2a-3i=10+bi$ b. $3a-8+5bi=2b-ai-3i$

**5)** Solve the following equation: $2z+7i=iz+\overline{z}-3$.

**6)** Compute the following square roots:

a. $\sqrt{5-12i}$ b. $\sqrt{8+6i}$

**7)** Solve the following equation: $z^2-2(1-2i)z-8i=0$.

**8)** Solve the following equation: $iz^2-2(1-i)z+6+15i=0$.

**9)** Solve the following equation: $z^2-i\overline{z}+6=0$.

**10)** Given the following equation in $z$: $(mi-2)z^2-2(m+2i)z+1=0$.

For which values of the complex parameter $m$ does the equation:

a. have a single (unique) solution?
b. have no solution?

## Answer Key

**1)** $4-i$

**2)** $-4+5i$

**3)** $z=2-3i,\ w=5+i$

**4)** a. $a=5, b=-3$ b. $a=2, b=-1$

**5)** $z=-\frac{1}{2}-\frac{5}{2}i$

**6)** a. $\pm(3-2i)$ b. $\pm(3+i)$

**7)** $2,\ -4i$

**8)** $3i,\ -2-5i$

**9)** $2i,\ -3i$

**10)** a. m=I b. m=-2i

## The Complex Plane

## Questions

**1)** Write the following complex numbers in standard (rectangular) form:

a. $2(\cos 60° + i\sin 60°)$

b. $6(\cos 135° + i\sin 135°)$

c. $4(\cos 330° + i\sin 330°)$

d. $8(\cos(-30°) + i\sin(-30°))$

e. $4(\cos 690° + i\sin 690°)$

f. $8(\cos 90° + i\sin 90°)$

g. $3(\cos 270° + i\sin 270°)$

h. $\cos 180° + i\sin 180°$

i. $\cos 0° + i\sin 0°$

**2)** Write the following complex numbers in polar form:

a. $1+i$

b. $\sqrt{3}-i$

c. $-\frac{1}{2}-\frac{\sqrt{3}}{2}i$

d. $3+4i$

e. $6i$

f. $-i$

g. $4$

h. $-1$

i. $1$

j. $0$

**3)** Compute the following expressions in polar form:

a. $2(\cos 120° + i\sin 120°)\cdot 3(\cos 60° + i\sin 60°)$

b. $(\cos 210° + i\sin 210°)\cdot 5(\cos(-40°) + i\sin(-40°))$

c. $\dfrac{12(\cos 315° + i\sin 315°)}{3(\cos 90° + i\sin 90°)}$

d. $\dfrac{1}{2(\cos 40° + i\sin 40°)}$

e. $6(\cos 30° + i\sin 30°) + 2(\cos 210° + i\sin 210°)$

**4)** Given the complex number $z = r(\cos\theta + i\sin\theta)$.
Express the following in terms of $r$ and $\theta$:

a. $\overline{z}$ b. $1/z$ c. $-z$

d. $-\frac{1}{z}$ e. $iz$ f. $z \cdot \overline{z}$

**5)** Let *z* be any complex number. Show that the following are purely real numbers:

a. $z + \overline{z}$ b. $z \cdot \overline{z}$ c. $\frac{z}{\overline{z}} + \frac{\overline{z}}{z}$

**6)** Let *z* be any complex number. Show that the following are purely imaginary numbers:

a. $z^2 - \overline{z}^2$ b. $\frac{1}{\overline{z}} - \frac{1}{z}$

**7)** Prove the following identities:

a. $z - i\overline{z} = \overline{\overline{z} + iz}$ b. $z \cdot \overline{z} = |z|^2$

**8)** A complex number $w$ is located outside of the unit circle in the complex plane.
Decide if each of the following numbers is inside, outside or on the unit circle:

a. $\overline{w}$ b. $1/w$ c. $w/\overline{w}$ d. $w \cdot \overline{w}$

**9)** A square, whose sides are parallel to the axes, is inscribed in the canonical circle of radius $\sqrt{2}$ in the complex plane. Find its vertices.

**10)** A square, is inscribed in a canonical circle in the complex plane.
Find its vertices, given that one of them is $1+\sqrt{3}i$.

**11)** An equilateral triangle, is inscribed in a canonical circle in the complex plane.
Find its vertices, given that one of them is $1+\sqrt{3}i$.

**12)** An isosceles triangle, whose base angle is $30°$ is inscribed in a canonical circle in the complex plane. The apex is at $1+\sqrt{3}i$. Find the other vertices.

## Answer Key

**1)** a. $1+\sqrt{3}i$ b. $-3\sqrt{2}+3\sqrt{2}i$ c. $2\sqrt{3}-2i$ d. $4\sqrt{3}-4i$ e. $2\sqrt{3}-2i$
f. $8i$ g. $-3i$ h. $-1$ i. $1$

**2)** a. $\sqrt{2}\left(\cos 45°+i\sin 45°\right)$ b. $2\left(\cos 270°+i\sin 270°\right)$ c. $\cos 240°+i\sin 240°$
d. $5\left(\cos 53.13°+i\sin 53.13°\right)$ e. $6\left(\cos 90°+i\sin 90°\right)$ f. $\cos 270°+i\sin 270°$
g. $4\left(\cos 0°+i\sin 0°\right)$ h. $\cos 180°+i\sin 180°$ i. $0\left(\cos 0°+i\sin 0°\right)$
j. $0$

**3)** a. $6\left(\cos 180°+i\sin 180°\right)$ b. $5\left(\cos 170°+i\sin 170°\right)$ c. $4\left(\cos 225°+i\sin 225°\right)$
d. $\frac{1}{2}\left(\cos(-40°)+i\sin(-40°)\right)$ e. $4\left(\cos 30°+i\sin 30°\right)$

**4)** a. $r(\cos(\text{-}\theta))+i\sin(\text{-}\theta))$ or $r(\cos\theta-i\sin\theta)$
b. $\frac{1}{r}(\cos(\text{-}\theta))+i\sin(\text{-}\theta))$ or $\frac{1}{r}(\cos\theta-i\sin\theta)$
c. $-r(\cos(\theta))+i\sin(\theta))$ or $r\left(\cos(\theta+180°)+i\sin(\theta+180°)\right)$
d. $\frac{1}{r}(\cos(180°-\theta))+i\sin(180°-\theta))$ or $\frac{1}{r}(-\cos\theta+i\sin\theta)$
e. $r\left(\cos(\theta+90°)+i\sin(\theta+90°)\right)$
f. $r^2$ or $r^2\left(\cos 0°+i\sin 0°\right)$

**5)** a. $2a$ b. $a^2+b^2$ c. $2(a^2-b^2)$

**6)** a. $4ab$i b. $\frac{2b}{a^2+b^2}$i

**7)** Proof.

**8)** a. outside b. inside c. on d. outside

**9)** $1+i,\ -1+i,\ -1-i,\ 1-i$

**10)** $1+\sqrt{3}i,\ -\sqrt{3}+i,\ -1-\sqrt{3}i,\ \sqrt{3}-i$

**11)** $1+\sqrt{3}i,\ 2,\ 1-\sqrt{3}i$

**12)** $1+\sqrt{3}i,\ -1+\sqrt{3}i,\ 2$

## De Moivre's and N-th Root

## Questions

**1)** Use De Moivre's formula to evaluate the following expressions:

a. $\left[2\left(\cos 30° + i\sin 30°\right)\right]^3$ b. $\left[2\left(\cos 14° + i\sin 14°\right)\right]^5$ c. $\left(1+i\right)^4$

d. $\left(\sqrt{3}-i\right)^3$ e. $\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^{12}$

**2)** Solve the following equations:

a. $z^2 = 36\left(\cos 120° + i\sin 120°\right)$ b. $z^4 = \left(9\left(\cos 80° + i\sin 80°\right)\right)^2$ c. $z^5 = \frac{1}{2}+\frac{\sqrt{3}}{2}i$

**3)** Find the sum and the product of the 4th roots of unity.

**4)** Find the locus of points z in the complex plane such that $|z| = 2$.

**5)** Find the locus of points z in the complex plane such that $|z-3i| = 5$.

**6)** Find the locus of points z in the complex plane such that $|z+i| + |\bar{z}+i| = |1+3i|$.

## Answer Key

**1)** a. $8i$ b. $32(\cos 70° + i\sin 70°)$ c. $-4$ d. $-8i$ e. $1$

**2)** a. $z = 6(\cos 60° + i\sin 60°)$ or $6(\cos 240° + i\sin 240°)$

b. $z = 3(\cos 40° + i\sin 40°)$ or $3(\cos 130° + i\sin 130°)$ or $3(\cos 220° + i\sin 220°)$ or $3(\cos 310° + i\sin 310°)$

c. $z = \cos 12° + i\sin 12°$ or $\cos 84° + i\sin 84°$ or $\cos 156° + i\sin 156°$ or $\cos 228° + i\sin 228°$ or $\cos 300° + i\sin 300°$

**3)** sum: 0 , product: $-1$

**4)** $x^2 + y^2 = 4 = 2^2$ Canonical circle with radius 2

**5)** $x^2 + (y-3)^2 = 25 = 5^2$ Circle centred at (0,3) with radius 5.

**6)** $10x^2 + 6y^2 = 15$ (ellipse)

## Arithmetic and Geometric Sequences

## Questions

**1)** In a complex arithmetic sequence, the 3rd member is $a_3 = 5 - 9i$ and the 7th member, is $a_7 = 13 + 3i$. Find the sum of the first 10 members of the sequence.

**2)** In a complex geometric sequence, the 2nd member is $a_2 = 2 - 4i$ and the 5th member, is $a_5 = 32 + 16i$.

a. Find the first member $a_1$ and the quotient $q$, given that $q$ is purely imaginary.

b. Find the sum of the first 5 members of the sequence.

## Answer Key

**1)** $100-15i$

**2)** a. $a_1 = 2+i,\ q = -2i$ b. $20+25i$

## Miscellaneous Problems

## Questions

**1)** Let $(z_1, z_2, z_3)$ be a geometric sequence.
Given $z_1 = 2$ and given that if we add $4i$ to $z_3$ we get an arithmetic sequence, find the members of geometric sequence. [There are two possible answers.]

**2)** Solve the equation: $z - \overline{z} + |z| = |2-i|^2 - 4i + \mathrm{Im}(z)\cdot i$

**3)** Solve the equation: $\left|2 - 3^{x^2-x-1} i\right| = \sqrt{13}$

**4)** Solve the equation $z^3 = \overline{z}$ in two ways:
   a. Using rectangular form
   b. Using polar form

**5)** **Prove**: If $az^2 + bz + c = 0 \ (a \neq 0)$ is a quadratic equation with real coefficients, and if the equation has no real roots, then its roots are a pair of complex conjugates.

**6)** **Prove**: If $z_1$, $z_2$ are complex numbers which aren't purely real, and if $z_1 + z_2$ and $z_1 \cdot z_2$ are both real, then $z_1$, $z_2$ are conjugates.

**7)** **Prove**: If $z$ is a complex number which isn't purely real, and if $z - \frac{1}{\overline{z}}$ is purely real, then $z$ is on the unit circle.

**8)** **Prove** the formula for multiplication in polar form:
$r_1(\cos\theta_1 + i\sin\theta_1)\cdot r_2(\cos\theta_2 + i\sin\theta_2) = r_1 r_2\left(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right)$

**9)** Given a complex number *z* on the unit circle in the first quadrant, and given that $\left|z^4 - z^3\right| = \sqrt{2-\sqrt{3}}$, find $\arg(z)$.

**10)** Given a complex number *z* on the unit circle. Find the value of the expression $z + iz$, given that it's purely real. Solve twice:
   a. Using rectangular form.
   b. Using polar form.

**11)** Given that $z_1$ and $z_2$ are the solutions of the quadratic $z^2 - 2\cos\theta\cdot z + 1 = 0$, and let $P_1, P_2$ be their corresponding points in the complex plane.
Find the size of the angle $\sphericalangle P_1 O P_2$.

## Answer Key

**1)** $(2,\ 4-2i,\ 3-4i)$ or $(2,\ 2i,\ -2)$

**2)** $z=\pm 3-4i$

**3)** $x=2$ or $x=-1$

**4)** $z=0$ or $1$ or $i$ or $-1$ or $-i$

**5)** Proof

**6)** Proof

**7)** Proof

**8)** Proof

**9)** $\arg(z)=30^{\circ}$

**10)** $\pm\sqrt{2}$

**11)** $\sphericalangle P_1OP_2=2\theta$