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Workbook

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1) $z_n = (2n+1)\pi i$,Where n is any integer

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The Complex Sine Function

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1) Solve the equation $\sin z = 2$.

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Answer Key

1) $z=\left(\frac{\pi}{2}+2\pi n\right)+i\cdot\ln\left(2\pm\sqrt{3}\right)$, where *n* is any integer.

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The Complex Cosine Function

Questions

1) Solve the equation $\cos z = 2$.

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Answer Key

1) $z = 2\pi n + i \cdot \ln\left(2 \pm \sqrt{3}\right)$, where *n* is any integer.

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Elementary Transformations

Questions

1) Consider the transformation $f(z) = z+1$ and define the domain $U = \{z \in \mathbb{C} \mid |z| < 1\}$.
Find the image $f(U)$.

2) Consider the transformation $f(z) = 3z$ and define the domain $U = \{z \in \mathbb{C} \mid |z| < 1\}$.
Find the image $f(U)$.

3) Consider the transformation $f(z) = \frac{-1+i}{\sqrt{2}} \cdot z$ and define the domain $U = \{z \in \mathbb{C} \mid \arg(z) = \frac{\pi}{4}\}$.
Find the image $f(U)$.

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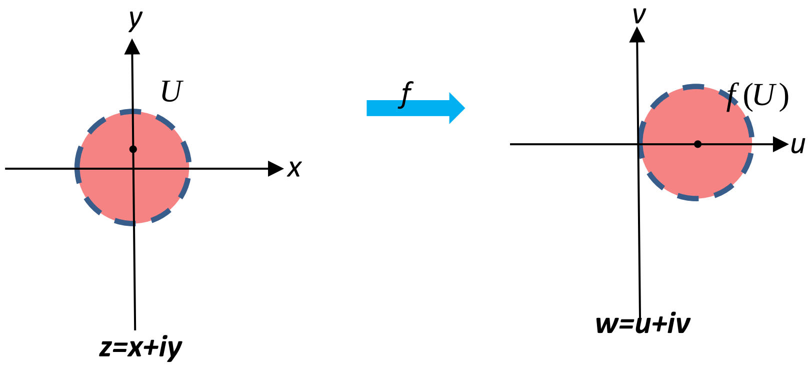
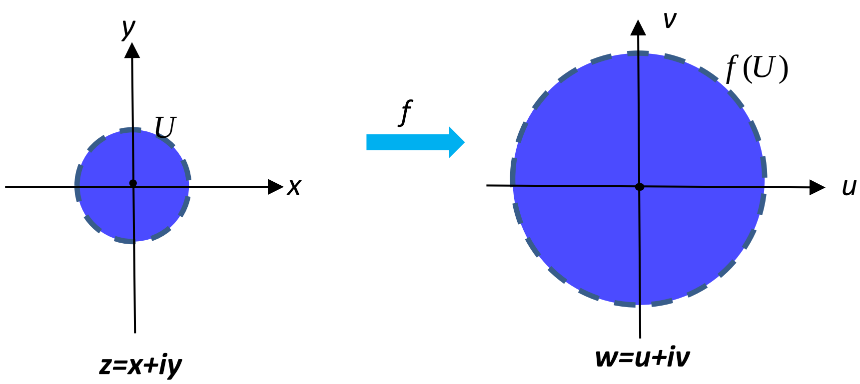
Answer Key

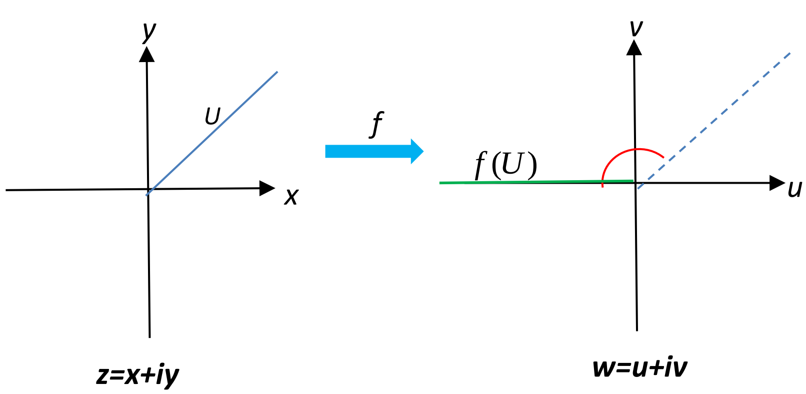
1) $f(U)=\{w\in\mathbb{C}\mid |w-1|<1\}$ circle with radius 1 and center $(1,0)$



2) $f(U)=\{w\in\mathbb{C}\mid |w-1|<1\}$ circle with radius 3 and center $(0,0)$



3) $f(U)=\{w\in\mathbb{C}\mid \arg(w)=\pi\}$



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Complex Logarithms, Multivalued Functions, Roots

Questions

1) Find all possible values of $\sqrt{i}$.

2) Find all possible values of i^i.

3) Find all possible values of $2^{\frac{1}{9}+\frac{i}{50}}$.

4) Find any 3 values of $(2+2i)^{5i}$. How many are there altogether?

5) The following are 4 rules of exponents, where $c,d\in\mathbb{C}$; $z\in\mathbb{C}^\times$; $n\in\mathbb{Z}$

a. $z^{-c}=\frac{1}{z^c}$

b. $z^{c+d}=z^c\cdot z^d$

c. $z^{c-d}=\frac{z^c}{z^d}$

d. $\left(z^c\right)^n=z^{c\cdot n}$

Show that the 4th rule doesn't always hold if *n* is not an integer. Hint: compute $\left(i^2\right)^i$.

6) Given the function $F:\hat{\mathbb{C}}\to\mathbb{C}$, $F(z)=\mathrm{Log}\left(\frac{z-a}{z-b}\right)$, where $a,b\in\mathbb{C}$ and $a\neq b$.

Find a domain $U\subseteq\hat{\mathbb{C}}$, where *F* is analytic. [Assumes knowledge of Möbius ransformations.]

7) Show that if $a>0$ then $\left|a^z\right|=a^{\mathrm{Re}\,z}$.

8) One way we could define the square root on $\mathbb{C}^\times$ is as follows:

if $z\neq0$ then $z=re^{i\theta}$ for a unique $\theta\in[0,2\pi)$, and then let $\sqrt{z}=\sqrt{r}e^{i\frac{\theta}{2}}$.

Note: this is not the principal value, where $\theta\in(-\pi,\pi]$.

Prove that this definition of $\sqrt{z}$ is not continuous.

9) Find all possible values of $\log\left(\log(-1)\right)$.

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10) Given: $\log(z)$ is a branch of $\log z$ on $\mathbb{C}-\{z=x+iy \mid x\geq 0, y=\sin x\}$ and $\log(1)=0$.

Compute the following values: $\log(-1), \quad \log i, \quad \log(-i), \quad \log\frac{5\pi}{2}, \quad \log\frac{3\pi}{2}$.

11) Let's define $\log_{\alpha} z=\ln|z|+i\cdot\arg z$ where we choose (uniquely) .

a. Com $\alpha-2\pi<\arg z\leq\alpha$ pute the following values: $\log_0 1,\ \log_{\pi} 1,\ \log_{2\pi} 1$.

b. Find the image of $\mathbb{C}-(-\infty,0]$ under the transformation $z\mapsto\log_{\pi} z$.

c. Find the image of $\mathbb{C}-[0,\infty)$ under the transformation $z\mapsto\log_0 z$.

12) Let $z_1,z_2,\ldots z_n\in\mathbb{C}$ be such that $\operatorname{Re} z_k>0$ for $1\leq k\leq n$ and $\operatorname{Re}(z_1\cdot z_2\cdot\ldots\cdot z_k)>0$ for $1\leq k\leq n-1$.

Prove: $\operatorname{Log}(z_1,z_2,\ldots z_n)=\operatorname{Log} z_1+\ldots+\operatorname{Log} z_n$. [Log is the principal value of log.]

13) Let $\operatorname{Log} z$ be the principal branch of $\log z$ on $U=\mathbb{C}-(-\infty,0]$.

Compute $\lim\limits_{y\to 0^+}\left[\operatorname{Log}(a+iy)-\operatorname{Log}(a-iy)\right]$ for $a>0$ and for $a<0$.

14) Let $D\subseteq\mathbb{C}$ be a domain containing 0. Prove that there's no analytic branch of $\sqrt[n]{z}$ in *D* $(n\geq 2)$. That is, there doesn't exist an analytic *f* on *D* such that $f(z)^n=z$ for all $z\in D$.

15) Let $U\subseteq\mathbb{C}^{\times}$ be a domain (open and connected).
Let $f(z)$ and $g(z)$ be two continuous branches of $\log z$ on *U*.
Prove that $\exists k\in\mathbb{Z}$ such that $f-g=2k\pi i$. I.e. $f(z)-g(z)=2k\pi i \quad \forall z\in U$.

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Answer Key

1) $z=\pm\left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)$

2) $\ldots, e^{\pi\frac{3}{2}}, e^{-\frac{1}{2}\pi}, e^{-\pi\frac{5}{2}}, e^{-\pi\frac{9}{2}}, \ldots$

3) $2^{\frac{1}{9}}e^{-\frac{2\pi k}{50}}e^{i\left(\frac{\ln(2)}{50}+\frac{2\pi k}{9}\right)}$, $k\in\mathbb{Z}$

4) $e^{-5\left(\frac{\pi}{4}+2\pi k\right)}e^{i\cdot 5\ln\sqrt{8}}$: $\underbrace{e^{-5\left(\frac{\pi}{4}-2\pi\right)}e^{i\cdot 5\ln\left(\sqrt{8}\right)}}_{k=-1}, \underbrace{e^{-5\cdot\frac{\pi}{4}}e^{i\cdot 5\ln\left(\sqrt{8}\right)}}_{k=0}, \underbrace{e^{-5\left(\frac{\pi}{4}+2\pi\right)}e^{i\cdot 5\ln\left(\sqrt{8}\right)}}_{k=1}$ Infinitely many Answers.

5) Proof

6) $U=\hat{\mathbb{C}}-[a,b]$

7) Proof

8) Proof

9) $\log\left(\log(-1)\right)=\begin{cases}\ln(\pi+2\pi k)\ +i\cdot\left(\frac{\pi}{2}+2\pi m\right) & k,m\in\mathbb{Z}\quad k\geq 0\\ \ln(-\pi-2\pi k)+i\cdot\left(-\frac{\pi}{2}+2\pi m\right) & k,m\in\mathbb{Z}\quad k<0\end{cases}$ OR

10) $\log(-1)=-\pi i$, $\log(i)=-\frac{3\pi}{2}i$, $\log(-i)=-\frac{\pi}{2}i$, $\log\frac{5\pi}{2}=\ln\frac{5\pi}{2}$, $\log\frac{3\pi}{2}=\ln\frac{3\pi}{2}-2\pi i$

11) a. $\log_0 1=0$, $\log_\pi 1=0$, $\log_{2\pi} 1=2\pi i$ b. $\{w\,|-\pi<\operatorname{Im} w<\pi\}$ c. $\{w\,|-2\pi<\operatorname{Im} w<0\}$

12) Proof

13) $a>0:\lim=0$, $a<0:\lim=2\pi i$

14) Proof

15) Proof

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