proprep

Workbook

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Properties of Analytic Functions

Liouville's Theorem

Questions

1) Find an entire function *f* which satisfies the inequality $|\sin z - zf(z)| < 2$ for all $z \in \mathbb{C}$.
Hint: use Liouville's Theorem.

2) Find an entire function *f* which satisfies $|\sin^2 z \cdot \cos z - z^2 \cdot f(z)| < 100$ for all $z \in \mathbb{C}$.

3) Find an entire function *f* which satisfies $\left| z\cos z + \left(z - \frac{\pi}{2}\right) f(z) \right| < 200$ for all $z \in \mathbb{C}$.

4) Find all entire functions $f(z) = u + iv$ such that $u \le 0$ for all *z*.
Hint: consider the function $e^{f(z)}$.

5) Find all entire functions $f(z) = u + iv$ such that $u \ge 0$ for all *z*.
Hint: consider the function $e^{-f(z)}$.

6) Find all entire functions $f(z) = u + iv$ such that $v \ge 0$ for all *z*.
Hint: consider the function $e^{i \cdot f(z)}$.

7) Let $f(z)$ be an entire functions such that $f(z) = u + iv$ such that $|f(z)| \ge 1$ for all *z*.
Prove that *f* is constant.
Hint: consider the function $\frac{1}{f(z)}$.

8) Find all entire functions $f(z) = u + iv$ such that $v \le 0$ for all *z*.
Hint: consider the function $e^{-i \cdot f(z)}$.

9) Prove that if $f(z)$ is entire and satisfies $|f(z)| \le 4 + 5|z|^{\frac{4}{5}}$ for all *z* then *f* is constant.

10) Prove that if $f(z)$ is entire and satisfies $|f(z)| \ge e^{\mathrm{Re}(z)}\ \forall z \in \mathbb{C}$ then there is a constant $c \in \mathbb{C}$ such that $f(z) = ce^{z}\ \forall z \in \mathbb{C}$.

11) Prove that if $f(z)$ is entire and satisfies $f(0) = 0$ and $f(1) = 1$,
then there is a constant $c \in \mathbb{C}$ such that $f(c) \ge 2$.

12) Prove that if $f(z)$ is entire and satisfies $\lim_{z \to \infty} f(z) = 2$ then $f(z) \equiv 2$.

13) Prove that if $f(z) = u + iv$ is entire and satisfies $u \cdot v \ge 0\ \forall z \in \mathbb{C}$ then *f* is constant.
Hint: consider $g(z) = f^2(z)$ and then $h(z) = e^{ig(z)}$.

14) Prove that if $f(z) = u + iv$ is entire and satisfies $u \ge v\ \forall z \in \mathbb{C}$ then *f* is constant.
Hint: consider $g(z) = f(z) + i \cdot f(z)$ and then $h(z) = e^{-g(z)}$.

15) Prove no analytic function $f(z)$ in $\mathbb{C}^{\times} = \mathbb{C} - \{0\}$ can satisfy $|f(z)| \ge \frac{1}{\sqrt{|z|}}\ \forall z \in \mathbb{C}^{\times}$.
You may use Riemann's Extension Theorem: If $D \subseteq \mathbb{C}$ be is a domain containing z_0, and if $g(z)$ is analytic and bounded on $D - \{z_0\}$, then g extends uniquely to an analytic function $\tilde{g}$ on all of *D*.

16) Prove that if $f(z)$ is entire and $\forall z \in \mathbb{C}$, $\mathrm{Re}\, f(z) \le |f(z)|^2$ then $f(z) \equiv const$

17) Prove that if $f(z)$ is entire and not constant then the image $f(\mathbb{C})$ is dense in $\mathbb{C}$
Definition: a set $A \subseteq \mathbb{C}$ is called dense in $\mathbb{C}$ if $\forall z_0 \in \mathbb{C}$ and $\forall \varepsilon > 0$ $D(z_0, \varepsilon) \cap A \ne \emptyset$.

18) It is known that there exists an function $T : \mathbb{C} - (-\infty, 0] \to D(0,1)$ satisfying
$T'(z) \ne 0 \forall z$. [For example $T(z) = e^{-\sqrt{z}}$ where $\sqrt{z} \equiv e^{\frac{1}{2}\mathrm{Log}(z)}$].
Suppose that $f : \mathbb{C} \to \mathbb{C} - (-\infty, 0]$ is an entire function. Prove that *f* is constant.

19) Let $f(z)$ be an entire function satisfying $f(z+1) = f(z)$ and $f(z+i) = f(z)$ for all *z*. Prove that *f* is constant.

Answer Key

1) $f(z)=\begin{cases}\dfrac{\sin z}{z}; & z\neq 0\\ 1 \quad ; & z=0\end{cases}$

2) $f(z)=\begin{cases}\left[\dfrac{\sin z}{z}\right]^2\cos z; & z\neq 0\\ \quad 1 \quad ; & z=0\end{cases}$

3) $f(z)=\begin{cases}\dfrac{z\cos z}{\frac{\pi}{2}-z}; & z\neq \dfrac{\pi}{2}\\ \dfrac{\pi}{2} \quad ; & z=\dfrac{\pi}{2}\end{cases}$

4) $f(z)\equiv a+bi,\ a\leq 0$

5) $f(z)=a+bi,\ a\geq 0$

6) $f(z)=a+bi,\ b\geq 0$

7) Proof

8) $f(z)=a+bi,\ b\leq 0$

9) Proof

10) Proof

11) Proof

12) Proof

13) Proof

14) Proof

15) Proof

16) Proof

17) Proof

18) Proof

19) Proof

The Identity Theorem

Questions

1) Prove that if $f(z)$ is entire and satisfies $f\left(\frac{1}{n}\right)=\frac{1}{n}\ \forall n\in\mathbb{N}$, then $f(z)\equiv z$. $\mathbb{N}=\{1,2,3,\dots\}$

2) Suppose $f(z)$ is analytic on $D=D(0,1)$ and satisfies $f\left(\frac{1}{n}\right)=\frac{1}{n+1}\ \forall n\in\mathbb{N}$. Find $f(z)$.

3) Suppose $f(z)$ is analytic on $D=D(0,1.5)$ and satisfies $f\left(\frac{1}{2n}\right)=\frac{1}{2\pi i}\oint_{|z|=1}\frac{nz}{nz-0.5}dz\ \forall n\in\mathbb{N}$.

Find $f(z)$.

4) Suppose $f(z)$ is analytic on $D=D(0,1)$ and satisfies $f\left(\frac{1}{3n-1}\right)=\frac{2}{n}\ \forall n\in\mathbb{N}$. Find $f(z)$.

5) Prove that if $f(z)$ is analytic on $D=D(0,1)$ and satisfies $f\left(\frac{1}{n}\right)=\sin(\pi n)\ \forall n\in\mathbb{N}$

then $f(z)\equiv 0$ in *D*.

6) Find all analytic functions $f(z)$ on $D=D(0,1)$ which satisfy

$$f\left(\frac{1}{n}\right)=\begin{cases}\frac{1}{n+1}\ ; & n=2k \quad (n \text{ even})\\ \frac{1}{n+2}\ ; & n=2k-1\ (n \text{ odd})\end{cases}$$

7) Find an analytic function $f(z)$ on $D=D(0,1)$ which has infinitely many zeros in *D*, or prove than no such f exists. Recall: $D(0,1)=\{z\,|\,|z|<1\}$ (open unit disk).

8) Is there an analytic function $f(z)$ on $D=\{z\,|\,1<|z|<3\}$ such that $f(x)=|x|^3$ for all real *x*, such that $1<|x|<3$? Find such an *f* or prove than no such *f* exists.

9) a. Prove the following theorem:
If $f(z), g(z)$ are analytic on a domain *D* and $f(z)g(z)\equiv 0$, then $f(z)\equiv 0$ or $g(z)\equiv 0$.
b. Let $f(z), g(z)$ be analytic in a domain *D* and let $a,b\in\mathbb{C}$ (arbitrary).
Prove that if $(f(z)-a)(g(z)-b)\equiv 0\ \forall z\in D$ then $f(z)\equiv a$ or $g(z)\equiv b$.

10) Note: This exercise assumes knowledge of the Residue Theorem.
Suppose that $f(z)$ is analytic on $D=D(z_0,R)$, where $z_0\in\mathbb{C}, R>0$,
and that $f'(z_0)\neq 0$. Prove that $\exists r$ such that $0<r<R$ $\frac{2\pi i}{f'(z_0)}=\oint_{|z-z_0|=r}\frac{1}{f(z)-f(z_0)}dz$

11) Note: This exercises assumes knowledge of Poles [a kind of Isolated Singular Point].
Define the domain $D=\left\{z \mid 0<|z|<1\right\}$, the punctured (open) unit disk Suppose that:

$f(z)$ is analytic in *D*, has a pole at $z=0$, and $f\left(\frac{1}{n}\right)=n \ \ \forall n \in \mathbb{N}$,

Prove that $f(z)=\frac{1}{z} \ \forall z \in D$.

12) Given an analytic function $f(z)$ on $|z|>1$ such that for all real $x>1$, $f(x)$ is real.
Prove that, for all real $x<-1$, $f(x)$ is real.

13) Given a continuous function $f(z)$ on $D=D(0,1)$ which satisfies $f\left(\frac{1}{n}\right)=\frac{1}{n^2} \ \forall n \in \mathbb{N}$,

and $f\left(\frac{i}{2}\right)=0$. Prove that *f* is not analytic on *D*.

14) $f(z)$ is analytic on $D=D(0,1)$, continuous on $\overline{D}$, and satisfies $|f(z)|=1 \forall z \in \partial D$.
Notation: $D:|z|<1,\ \bar{D}:|z|\le 1,\ \partial D=1$
Prove that *f* can have only finitely many zeros in *D*.
Hint: use the Bolzano-Weierstrass theorem:
Every bounded sequence has a convergent subsequence.

Answer Key

1) Proof

2) Proof

3) $f(z)=\frac{z}{1+z}$

4) $f(z)=z$

5) $f(z)=6\frac{z}{z+1}$

6) Proof

7) No such f exists.

8) $f(z)=sin\left(\frac{\pi}{1-z}\right)$, for example.

9) No such f exists.

10) Proof

11) Proof

12) Proof

13) Proof

14) Proof

The Maximum and Minimum Modulus Principles

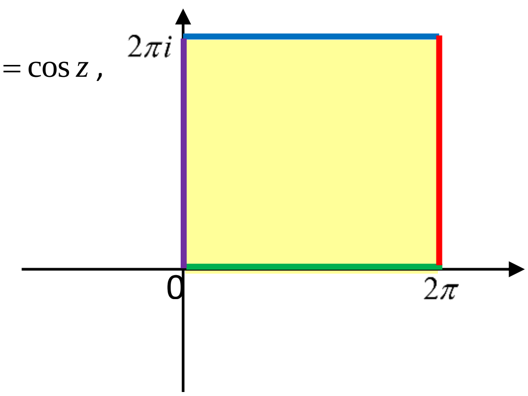
Questions

1) Let $f(z)=e^{-z^2}$ in $D=D(0,1)$. Does $|f|$ have a maximum value in *D*? If so, find it.

2) Let $f(z)=e^{-z^2}$ and let $D=D(0,3)$. Does *f* have a maximum value in $\bar{D}$? If so, find it.

3) True or False? If $f(z)$ is bounded on the domain $|z|>1$, it must be constant.

4) Find the maximum value (if it exists) of $|f(z)|$, where $f(z)=\cos z$, on the set $K=\{z=x+iy \mid 0\le x\le 2\pi,\ 0\le y\le 2\pi\}$.



5) Let $D=D(0,1)$ and let $f(z)=e^{z^2}$ in $\bar{D}$. Does *f* have a maximum value in $\bar{D}$? If so, find it and where it occurs.

6) Let f be analytic on $|z|<R$ and continuous on $K:|z|\le R$ such that $|f(z)|>a$ on $\partial K:|z|=R$ and $|f(0)|<a$, for some $a>0$. Prove that f has at least one zero in $|z|<R$.

7) Prove:

a. If $f(z)$ is a nonconstant analytic function on an open set *D* and $f(z)\neq 0$, then $|f(z)|$ can't have a local minimum in *D*.

b. Let $f(z)$ be an analytic function on an open set *D* such that $f(z)\neq 0$.

c. If $|f(z)|$ has a local minimum in *D*, then $f(z)$ is a constant function.

d. If $f(z)$ is analytic on a bounded open set *D*, and is nonzero and continuous on $\bar{D}$, then $|f(z)|$ achieves its minimum on ∂D.

8) Let $f(z)$ be analytic and <u>nonzero</u> on $K:|z|\le 1$ and suppose that $|f(z)|=1$ on $\partial K:|z|=1$.

a. Prove that $f(z)$ is constant.

b. Is the conclusion still true if we drop the requirement that $f(z)$ be nonzero?

9) Let $f(z)=u+iv$ be analytic on a bounded open set *D*, and continuous on $\bar{D}$.
Prove that $u(x,y)$ attains its maximum on ∂D.

10) Let $f(z)=u+iv$ be analytic on a bounded open set *D*, and continuous on $\bar{D}$.
Prove that $u(x,y)$ attains its minimum on ∂D.

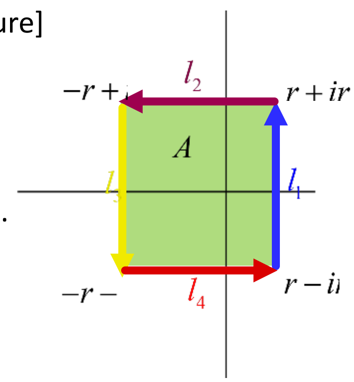
11) Let $f(z),\ g(z)$ be analytic on $D:|z|<1$, and continuous on $\bar{D}:|z|\le 1$,
for all z in $\partial D: |z| = 1$ we have $\operatorname{Re}\{f(z)\}=\operatorname{Re}\{g(z)\}$.
Prove: $f(z)\equiv g(z)+i\cdot c$ on $\bar{D}$ where *c* is a real constant.

12) Let $p(z)=\sum_{k=0}^{n} a_k z^k$ be a polynomial of degree $n\ (a_n\ne 0)$ such that $|p(z)|=1$ on $|z|=1$.
Prove that $|p(z)|\le |z|^n$ on $|z|\ge 1$.
Hint: show that $f(z)=z^n p\left(\frac{1}{z}\right)$ on $\mathbb{C}^{\times}$ can be extended to an entire function.

13) Let $f(z)$ be continuous on the compact set $A=\{z=x+iy\,|\,-r\le x\le r,\ -r\le y\le r\}\ (r>0)$,
and analytic on A°. We can write $\partial A=l_1\cup l_2\cup l_3\cup l_4$, where [see picture]

$l_1=\{r+iy\,|\,-r\le y\le r\}$ $\qquad$ $l_3=\{-r+iy\,|\,-r\le y\le r\}$

$l_2=\{x+ir\,|\,-r\le x\le r\}$ $\qquad$ $l_4=\{x-ir\,|\,-r\le x\le r\}$



Prove that $|f(0)|\le \frac{1}{4}\left(\max_{z\in l_1}|f(z)|+\max_{z\in l_2}|f(z)|+\max_{z\in l_3}|f(z)|+\max_{z\in l_4}|f(z)|\right)$.

Hint: $g(z)=\frac{1}{4}[f(z)+f(-z)+f(iz)+f(-iz)]$

14) Prove that if $A\subseteq\mathbb{C}$ is open and $f(z)$ is analytic and nonconstant on *A* then $f(A)$ is open.

15) Let $D=D(0,1)=\{z\in\mathbb{C}\,|\,|z|<1\}$ and let *f* be a function such that:

a. *f* is analytic on *D*
b. $|f(z)|\le 1$ for all $z\in D$
c. $f(0)=0$

Prove that $|f(z)|\le|z|\ \forall z\in D$ and that $|f'(0)|\le 1$.

16) Suppose that $f(z)$ is analytic on $\overline{D(0,1)}=\{z\in\mathbb{C} \mid |z|\le 1\}$ and $|f(z)|=1$ when $|z|=1$.

Suppose, in addition, that $f(z)=0 \Leftrightarrow z=0$.

Prove that $f(z)\equiv c\cdot z^k$ for some $c\in\mathbb{C}$ and $k\in\mathbb{N}$.

17) Suppose $f(z)$ is analytic on the annulus $A=\{z\in\mathbb{C}:1<|z|<2\}$ and continuous on $\bar{A}$.

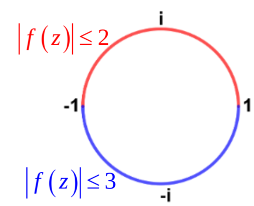
Suppose, in addition, that $|f(z)|=1$ on $|z|=1$ and that $|f(z)|=8$ on $|z|=2$.

Prove that $|f(z)|\le |z|^3$ for all $z\in A$.

18) Suppose $f(z)$ is analytic on $|z|\le 1$ and satisfies:

a. $|f(z)|\le 2 \text{ on } \{z \mid \operatorname{Im} z\ge 0 \text{ and } |z|=1\}$

b. $|f(z)|\le 3 \text{ on } \{z \mid \operatorname{Im} z\le 0 \text{ and } |z|=1\}$



Prove that $|f(0)|\le\sqrt{6}$.

Hint: consider $g(z)=f(z)\cdot f(-z)$.

19) Let $f(z)$ be analytic and nonzero $D=D(0,1)$.

Prove that there exists a sequence $\{z_n\}\subseteq D$ such that: $|z_n|\to 1$ and $\{f(z_n)\}$ is bounded.

Answer Key

1) Proof
2) No maximum.
3) e^9
4) False. Example, $f(z)=\frac{1}{z}$.
5) $\cosh 2\pi \approx 268$
6) Maximum is *e*. Occurs at $z=\pm 1$.
7) Proof
8) Proof
9) a. Proof b. No
10) Proof
11) Proof
12) Proof
13) Proof
14) Proof
15) Proof
16) Proof
17) Proof
18) Proof
19) Proof