proprep™

Workbook

Analytic Functions

Analytic Functions

Questions

1) Express the complex function $f(z) = z \cdot \text{Re}(z)$
in the form $f(\underbrace{x+iy}_{z}) = u(x,y) + i \cdot v(x,y)$

2) Express the complex function $f(z) = |z|^2$
in the form $f(\underbrace{x+iy}_{z}) = u(x,y) + i \cdot v(x,y)$

3) Express the complex function $f(z) = 2|z|^2 + i(\bar{z})^2$
in the form $f(\underbrace{x+iy}_{z}) = u(x,y) + i \cdot v(x,y)$

4) Express the complex function $f(z) = \dfrac{z}{1+|z|^2}$
in the form $f(\underbrace{x+iy}_{z}) = u(x,y) + i \cdot v(x,y)$

5) Express the complex function $f(z) = z^2 + \bar{z}$
in the form $f(\underbrace{x+iy}_{z}) = u(x,y) + i \cdot v(x,y)$

6) Express the complex function $f(\underbrace{x+iy}_{z}) = \dfrac{x^3}{3} - i \cdot \dfrac{y^3}{3}$
in the form $f(z) = \text{expression in } z$.

7) Evaluate the limit $\lim_{z \to 0} \dfrac{\bar{z}}{z} = ?$ if it exists.

8) Evaluate the limit $\lim_{z \to 0} \dfrac{z^4}{|z|^4} = ?$ if it exists.

9) Given the function $f(z) = \bar{z}$, $z \in \mathbb{C}$.
At which points, if any, is *f* differentiable?

10) Given the function $f(z) = \text{Re}(z)$, $z \in \mathbb{C}$.
At which points, if any, is *f* differentiable?

11) Given the function $f(z)=|z|^2,\ z\in\mathbb{C}$.
At which points, if any, is *f* differentiable?

12) Show that the function $f(z)=z^2+\mathrm{Im}(z)$ is not differentiable anywhere.

13) Show that the function $f(z)=f(x+iy)=xy+i(x^2+y^2)$ is differentiable only at $z=0$.

14) Find real values for the parameters a,b such that the function
$f(z)=e^{ax}\cos(3y)-ie^{-3x}\sin(by)$ is differentiable everywhere.

15) It is known that the function $f(z)=\dfrac{z}{\overline{z}}$ is not continuous at $z=0$.
Find all points *z* (if any) where *f* is differentiable.

16) Let $f(z)$ be differentiable in a domain *D* and suppose that $\mathrm{Re}\{f(z)\}=0\ \forall z\in D$.
Prove that $f(z)$ is constant.

17) Let $f(z)$ be differentiable and not constant in a domain *D*.
Define $g(z)=\overline{f(z)}\ \forall z\in D$.
Prove that $g(z)$ can't be differentiable everywhere on *D*.

18) Let $f(z)$ be piecewise defined as follows: $f(z)=\begin{cases} e^{-\frac{1}{z^4}} & ; z\neq 0 \\ 0 & ; z=0 \end{cases}$

a) Prove that *f* is discontinuous at $z=0$

b) Show that the C-R equations hold at $z=0$

19) Let $f(z)$ be analytic on the domain $H^+=\{z\in\mathbb{C}\mid \mathrm{Im}(z)>0\}$ [upper half-plane].
Prove: $g(z)=\overline{f(\overline{z})}$ is analytic on $H^-=\{z\in\mathbb{C}\mid \mathrm{Im}(z)<0\}$ [lower half-plane].

20) Show that $f(z)=e^{\mathrm{Re}(z)}$ is not differentiable anywhere.

21) Given the function $f\left(z\right)=cx^2-xy+ixy^2$ where *c* is a complex constant.

a) Given that *f* is differentiable at $z=i+1$, find the value of *c*.

b) Using this *c*, find all points at which *f* is differentiable.

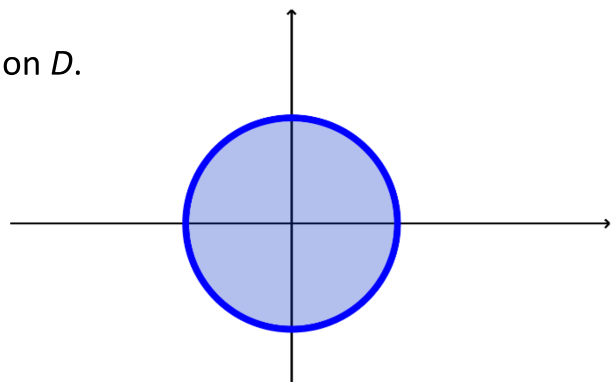
22) The function $f\left(z\right)=\frac{1}{2}\ln\left(x^2+y^2\right)+i\cdot\arctan\left(\frac{y}{x}\right)$ is defined on the right half-plane $H=\left\{z\in\mathbb{C}\ |\ \mathrm{Re}\left(z\right)>0\right\}$. Is *f* analytic on *H*?

23) Given the function $f\left(z\right)=e^{\frac{x^2-y^2}{2}}\left[\cos\left(xy\right)+i\cdot a\sin\left(xy\right)\right]$ where *a* is a real parameter. For which values of *a* is *f* holomorphic in the whole plane?

24)

Suppose the function $g\left(z\right)$ is holomorphic on the closed unit disk

$D=\overline{D\left(0,1\right)}=\left\{z\in\mathbb{C}\ |\ |z|\le 1\right\}$ and satisfies $\left|g\left(z\right)\right|=1$ on *D*.
Prove that *g* is constant on *D*.
Hint: express *g* as $g(z)=e^{i\cdot h(x,y)}$



25)

Let $D=D\left(0,R\right)=\left\{z\in\mathbb{C}\ |\ |z|<R,\ R>0\right\}$ and let $f:D\to\mathbb{C}$ be analytic on all of *D*.

Let $g\left(z\right)=\overline{f\left(\frac{R}{z}\right)}$. Find a suitable domain *A* for *g*, and check if *g* is differentiable on *A*.

26. Show that the function $u(x,y)=x^3-3xy^2$ is harmonic in the whole plane.

27. Show that the function $u(x,y)=x^2-y^2$ is harmonic in the whole plane and find a harmonic conjugate *v* of *u*.

28. Show that the function $f(z)=xy+i(x^2+y^2)$ is differentiable at the origin *O* but that its imaginary part is not harmonic at *O*. Is $f(z)$ holomorphic at *O?*

29. Show that the function $u(x,y)=\sin(x)\cosh(y)$ is harmonic in the entire plane and find its conjugate of $v(x,y)$ which satisfies $v(0,0)=2$.
Hint: consider the function $f(z)=\sin z$ * and use the formula
$\sin(x+iy)=\sin x\cosh y+i\cos x\sinh y$
*We'll study the complex sine in the next chapter.

30. Show that the function $u(x,y)=\cos(x)\sinh(y)$ is harmonic in the entire plane and find a holomorphic [analytic] $f(z)$ such that $u(x,y)=\text{Re}\{f(x+iy)\}$.
Hint: refer to the previous exercise.

31. Show that the function $v(x,y)=e^y\sin x$ is harmonic in the entire plane. Find a conjugate *u* of $-v$ * and an entire** function *f* such that $f(x+iy)=u(x,y)+iv(x,y)$.
* i.e. *v* is a conjugate of *u* ** holomorphic in the whole plane

32. Show that the polar function $u(r,\theta)=\left(r+\frac{1}{r}\right)\cos\theta$ is harmonic in for $r\neq 0$.

Reminder: In polar form, $u(r,\theta)$ is harmonic iff $r^2u''_{rr}+ru'_r+u''_{\theta\theta}=0$.

33. Given that the polar function $u(r,\theta)=\left(r+\frac{1}{r}\right)\cos\theta$ is harmonic in the domain $r\neq 0$, find a conjugate $v(r,\theta)$ of $u(r,\theta)$ in this domain.

34. Show that the function $u(x,y)=2x-x^3+3xy^2$ is harmonic and find a harmonic conjugate $v(x,y)$ of $u(x,y)$.

35. Let $f(\underbrace{x+iy}_{z})=u(x,y)+iv(x,y)$ be an entire function.

Prove that $g(x,y)=u(x,y)^2-v(x,y)^2$ is harmonic everywhere

36. Let $f(\underbrace{x+iy}_{z})=u(x,y)+iv(x,y)$ be an entire function.

Prove that $g(x,y)=\sin u(x,y)\cdot\cosh v(x,y)$ is harmonic everywhere.

37. Find (if any) all harmonic functions of the form $u(x,y)=\varphi\left(\frac{x^2+y^2}{x}\right)$ $(x\neq 0)$,

where $\varphi=\varphi(t)$ is a real-valued function with a continuous 2nd order derivative.

38. Find (if any) all harmonic functions of the form $u(x,y)=\varphi\left(\frac{x}{y}\right)$ $(y\neq 0)$,

where $\varphi=\varphi(t)$ is a real-valued function with a continuous 2nd order derivative.

Answer Key

1) $f(x+iy) = \underbrace{x^2}_{u} + i \cdot \underbrace{xy}_{v}$

2) $f(x+iy) = \underbrace{x^2+y^2}_{u} + i \cdot \underbrace{0}_{v}$

3) $f(x+iy) = \underbrace{2\left(x^2+xy+y^2\right)}_{u} + i \cdot \underbrace{\left(x^2-y^2\right)}_{v}$

4) $f(x+iy) = \underbrace{\frac{x}{1+x^2+y^2}}_{u} + i \cdot \underbrace{\frac{y}{1+x^2+y^2}}_{v}$

5) $f(x+iy) = \underbrace{x^2+x-y^2}_{u} + i \cdot \underbrace{(2xy-y)}_{v}$

6)

$$f(\underbrace{x+iy}_{z}) = \frac{x^3}{3} - i \cdot \frac{y^3}{3}$$

$$\begin{cases} z = x+iy \\ \bar{z} = x-iy \end{cases} \Rightarrow \begin{cases} x = \dfrac{z+\bar{z}}{2} \\ y = \dfrac{z-\bar{z}}{2i} \end{cases}$$

$$\boxed{(a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3}$$

$$f(z) = \frac{1}{3}\left[\frac{z+\bar{z}}{2}\right]^3 - \frac{1}{3} i \cdot \left[\frac{z-\bar{z}}{2i}\right]^3$$

$$= \frac{1}{3}\frac{z^3+3z^2\cdot\bar{z}+3z\cdot\bar{z}^2+\bar{z}^3}{2^3\ 8} - \frac{1}{3} i \cdot \frac{z^3-3z^2\cdot\bar{z}+3z\cdot\bar{z}^2-\bar{z}^3}{(2i)^3\ -8i}$$

$$= \frac{1}{24}\left[\left(z^3+3z^2\cdot\bar{z}+3z\cdot\bar{z}^2+\bar{z}^3\right)+\left(z^3-3z^2\cdot\bar{z}+3z\cdot\bar{z}^2-\bar{z}^3\right)\right]$$

$$= \frac{1}{24}\left[2z^3+6z\cdot\bar{z}^2\right]$$

$$f(z) = \frac{1}{12}\left(z^3+3z\cdot\bar{z}^2\right)$$

$$\text{or } \frac{z^3+3z\cdot\bar{z}^2}{12} \text{ or } \frac{1}{12}z^3+\frac{1}{4}z\cdot\bar{z}^2$$

7) The limit does not exist.

8) The limit does not exist.

9) $\varnothing$
f is nowhere differentiable.

10) $\varnothing$
f is nowhere differentiable.

11) Z = 0

12) No answer

13) No answer

14) $a = -3, b = 3$

15) $\varnothing$
f is differentiable nowhere!

16) No answer

17) No answer

18) No answer

19) No answer

20) No answer

21) $c = \frac{3}{2}$

$0, \quad 1+i, \quad 0.25+0.5i$

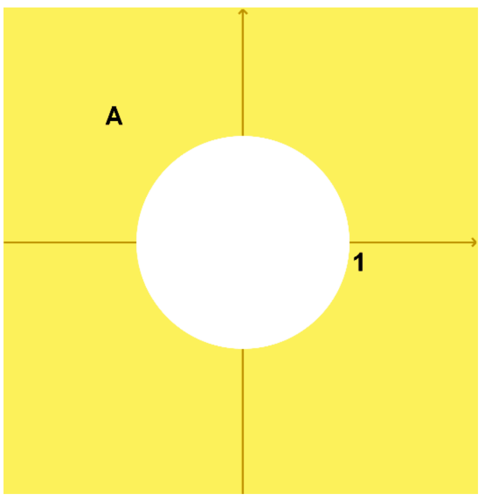
22) Yes

23) $a = 1$

24) No answer

25)

$A = \{ z \mid |z| > 1 \}$



26) No answer

27) Need to show that that Laplacian $\Delta u = u''_{xx} + u''_{yy} = 0$ in the whole plane:

$v(x, y) = 2xy$

28) No answer

29) $v(x, y) = \cos x \sinh y + 2$

30) $f(z) = -i \sin z$

31) $u(x,y) = -e^{y}\cos(x)$

$f(z) = -e^{-iz}$

32) No answer

33) $v(r,\theta) = \left(r - \frac{1}{r}\right)\sin\theta$

34) $v(x,y) = 2y - 3x^2y + y^3$

35) No answer

36) No answer

37) $u(x,y) = -\frac{c_1 \cdot x}{x^2 + y^2} + c_2 \quad (x \neq 0)$

38) $u(x,y) = c_1 \arctan\frac{x}{y} + c_2 \quad (y \neq 0)$