Workbook

Table of Contents

Applications of the Definite Integral – Area and Curve Length

Areas

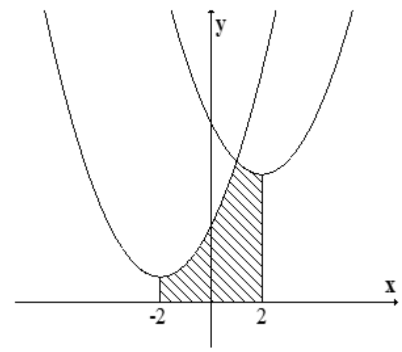
Questions

1) Given two functions:

$f(x) = x^2 + 4x + 6$, $g(x) = x^2 - 4x + 14$.

a. Find their point of intersection.

b. Find the area bounded by the graphs of the two functions, the x-axis, and the lines $x = 2$ and $x = -2$ (the shaded area in the figure).

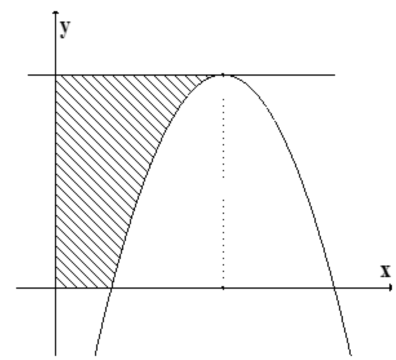


2) Given the function $y = -x^2 + 6x - 5$ (see figure).

a. Find the coordinates of the maximum point of the function.

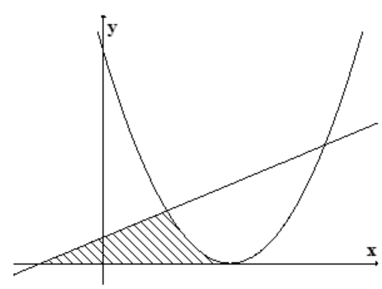
b. Find the equation of tangent to the graph at its maximum point.

c. Find the area bounded by this tangent, by the axes and by the graph of the function (the shaded area in the figure).



3) Given the function $f(x) = (x-2)^2$ and the line $y = 0.5x + 0.5$ (see figure).

Find the area bounded by the graph of the function, the given line and the x-axis (the shaded area in the figure).

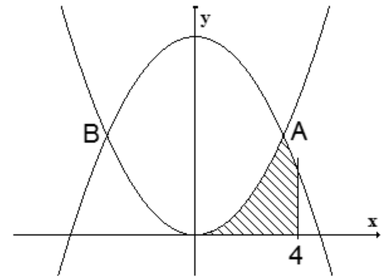


4) Given the functions: $f(x)=x^2$, $g(x)=-x^2+18$.

Their graphs intersect at points A and B (see figure).

a. Find the x-coordinates of the points A and B.

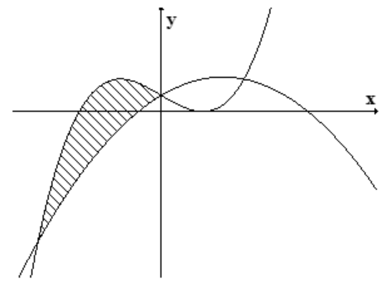
b. Compute the area, in the first quadrant, bounded by the functions, by the x-axis and by the line $x=4$ (The shaded area in the figure).



5) Given two functions: $y=-x^2+3x+2$, $y=x^3-3x+2$.

a. Find the x-coordinates of the intersection points of the graphs of the two functions.

b. Compute the area bounded by the graphs of the two functions (The shaded area in the figure).

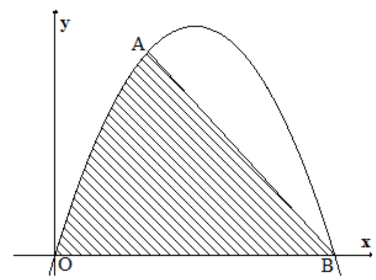


6) Given the function $f(x)=-x^2+ax$. The function passes through the point $\mathrm{A}(2,8)$ (see figure).

a. Find the parameter a.

b. The function cuts the x-axis at the origin and at another point B. Find the coordinates of B.

c. Compute the area (shaded in the figure) bounded by the graph of the function, by the chord AB and by the x-axis.



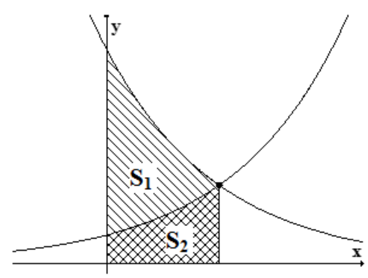
7) Given the functions: $f(x)=e^{-x+2}$, $g(x)=e^x$.

a. Find the intersection points of the functions with the y-axis.

b. Find the intersection point of the two functions.

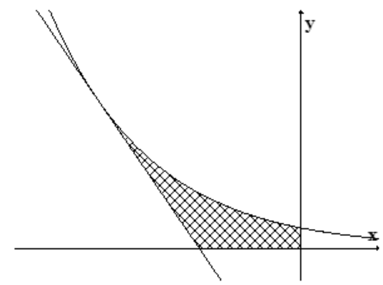
c. The areas S_1 and S_2 are as in the figure.

Compute the ratio $\frac{S_1}{S_2}$.



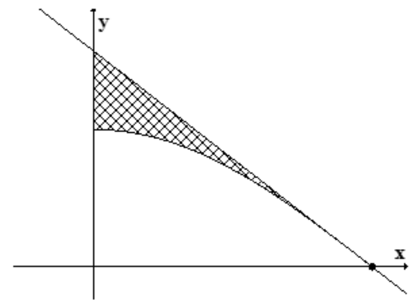
8) Given the function $f\left(x\right)=e^{-2x}$. A line to the function was drawn at the point $x=-1$ (see figure).

a. Find the equation of the tangent.
b. Compute the area (shaded in the figure) bounded by the graph of the function, by the tangent and by the axes.

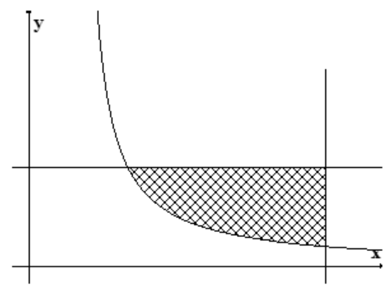


9) Given the function $y=\cos 2x$ on the domain $0\le x\le\frac{\pi}{4}$ (see figure). A tangent to the graph is drawn at the point, where $x=\frac{\pi}{4}$.

a. Find the equation of the tangent.
b. Compute the area (shaded in the figure) bounded by the graph of the function, by the tangent and by the y-axis.

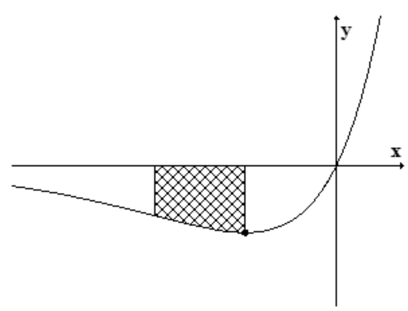


10) Compute the area bounded by the graph of the function $y=\frac{1}{2x-1}$, and by the lines $x=3$ and $y=1$. (The shaded area in the figure).



11) Given the function $f\left(x\right)=e^{2x}-e^{x}$, which has a minimum as shown in the figure.

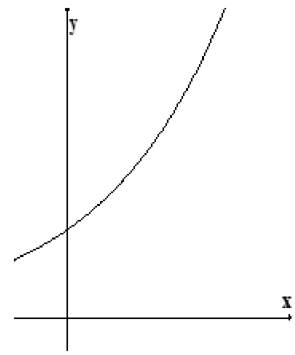
a. Find the x-coordinate of this minimum point.
b. A perpendicular is dropped from the minimum point to the x-axis. The shaded area in the figure is bounded by this perpendicular, the graph of the function, the x-axis and the line $x=a$.
   Given that it has an area of $3e^{2a}-e^{a}$, where $a<\ln 0.5$, find the value of a.



12) Given the function $f(x)=e^{\frac{x+1}{2}}$ as in the figure.

The slope of the tangent to the graph at a point A on it is $\frac{e^2}{2}$.

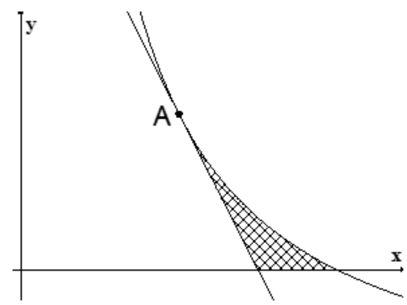
a. Find the coordinates of point A.
b. Find the equation of the tangent.
c. Compute the area bounded by: the graph of the function, the tangent and the y-axis.



13) Given the function $f(x)=\frac{8}{x}-2$ on the domain $x>0$.

A tangent line to the graph is drawn at the point $A(2,2)$, (see figure).

a. Find the equation of the tangent.
b. Compute the area bounded by: the graph of the function, the tangent and the x-axis.



14) Answer the following:

a. Draw a rough sketch of the graphs of the following 2 functions:

$$f(x)=\sin x \ ; \ 0\le x\le \pi$$
$$g(x)=\cos 2x \ ; \ 0\le x\le \pi$$

b. Shade or highlight the area bounded by these and compute its size.

15) Given the function $f(x)=\tan^2 x$ on the domain $-\frac{\pi}{2}<x\le 0$:

a. Find the equation of its tangent at the point where $x=-\frac{\pi}{4}$.
b. Show that $\int \tan^2 x dx=\tan x - x + c$, and find the area bounded by the graph of the function, the tangent and the x-axis.

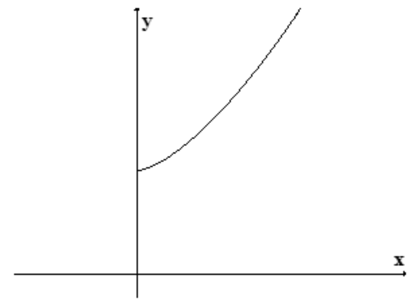
16) Through the point $A(8,0)$ tangents are drawn to the parabola $y=x^2-10x+25$.

a. Find the equations of the tangents.
b. Compute the area bounded by the tangents and the parabola.

17) Given the function $f(x) = x\sqrt{x} + 4$ on the domain $x \geq 0$ (see figure).

a. Find the equation of the line passing through the origin and tangent to the graph of the given function.

b. Compute the area bounded by the graph of the function, the tangent and the y-axis.



18) Answer the following questions:

a. Compute the derivative of the function $f(x) = \cos^3 x$.

b. Compute the area bounded the x-axis and by the graph of the function $y = \cos^2 x \cdot \sin x$ on the domain $\frac{1}{2}\pi \leq x \leq \frac{3}{2}\pi$.

19) Compute the area bounded by the parabola $y^2 = -x$ and the line $y = x + 6$.

20) Compute the area bounded by the parabola $x = y^2 + 2$ and the line $y = x - 8$.

21) Compute the following integrals:

a. $\int_0^a \sqrt{x^2 - a^2}\,dx$

b. $\int_{-a}^a \sqrt{a^2 - y^2}\,dy$

Answer Key

1) a. $(1,11)$ b. 25.33

2) a. $(3,4)$ b. $y=4$ c. $6\frac{2}{3}$

3) $1\frac{1}{3}$

4) a. $x=\pm 3$ b. $14\frac{2}{3}$

5) a. $x=0,-3,2$ b. $15\frac{3}{4}$

6) a. $a=6$ b. $\text{B}(6,0)$ c. $25\frac{1}{3}$

7) a. $(0,1),(0,e^2)$ b. $(1,e)$ c. $\frac{S_1}{S_2}=e-1$

8) a. $y=-2e^2x-e^2$ b. $\frac{1}{4}e^2-\frac{1}{2}$

9) a. $y=-2x+\frac{\pi}{2}$ b. $\frac{\pi^2}{16}-\frac{1}{2}$

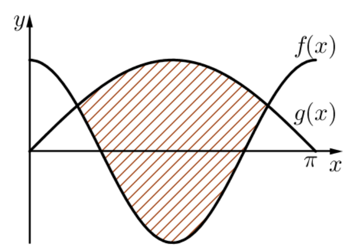
10) $2-\frac{1}{2}\ln 5$

11) a. $x=\ln 0.5$ b. $a=\frac{1}{2}\ln(0.15)$

12) a. $\text{A}(3,e^2)$ b. $y=\frac{e^2}{2}x-\frac{e^2}{2}$ c. $\frac{5}{4}e^2-2e^{0.5}$

13) a. $y=-2x+6$ b. $8\ln 2$ - 5

14) b. $\frac{3\sqrt{3}}{2}$ a. See the following sketch:



15) a. $y=-4x-\pi+1$ b. $\frac{7}{8}-\frac{\pi}{4}$

16) a. $y=0$ b. 18

17) a. $y=3x$ b. 4.8

18) a. $f'(x)=-3\cos^2 x\sin x$ b. $\frac{2}{3}$.

19) $20\frac{5}{6}$.

20) $20\frac{5}{6}$.

21) a. $0.25\pi a^2$ b. $0.5\pi a^2$.

Curve Length

Questions

1) Find the length of the curve $y = x^{\frac{2}{3}}$ between $x = 1$ and $x = 8$.

2) Find the length of the curve $y = \frac{x^4}{8} + \frac{1}{4x^2}$ from $x = 1$ to $x = 2$.

3) Find the length of the curve $y = \frac{x^5}{15} + \frac{1}{4x^3}$ from $x = 1$ to $x = 2$.

4) Find the length of the curve $y = \frac{2}{3}\left(1 + x^2\right)^{3/2}$ between $x = 0$ and $x = 3$.

5) Find the length of the curve $y = \frac{1}{3}\sqrt{x}\left(3 - x\right)$ between $x = 0$ and $x = 3$.

6) Find the length of the curve $y = \ln x$ from $\left(1,0\right)$ to $\left(2,\ln 2\right)$.

7) Find the length of the curve $y = x^2$ between $x = 1$ and $x = 2$.

You can use the formula: $\int \sqrt{x^2 \pm a^2}\,dx = \frac{1}{2}x\sqrt{x^2 \pm a^2} \pm \frac{1}{2}a^2 \ln\left|x + \sqrt{x^2 \pm a^2}\right|$.

8) Find the length of the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 4$ between $x = 1$ and $x = 8$.

9) Find the length of the curve $x = \frac{2}{3}y^{\frac{2}{3}}$ between $y = 0$ and $y = 3$.

10) Find the length of the curve $x = \frac{y^4}{4} + \frac{1}{8y^2}$ between $y = 1$ to $y = 2$.

11) Find the distance traveled between $t = 0$ and $t = \frac{\pi}{2}$ by a particle $\mathrm{P}\left(x, y\right)$, whose position at time t is given by: $x = \cos t + t\sin t,\ \ y = \sin t - t\cos t$.

12) Find the distance traveled between $t = 0$ and $t = 4$ by a particle $\mathrm{P}\left(x, y\right)$, whose position at time t is given by: $x = \frac{t^2}{2},\ \ y = \frac{1}{3}\left(2t + 1\right)^{3/2}$.

13) Find the distance traveled between $t = 0$ and $t = 4$ by a particle $\mathrm{P}\left(x, y\right)$, whose position at time t is given by: $x = \frac{1}{3}\left(2t + 3\right)^{3/2},\ \ y = \frac{t^2}{2} + t$.

Answer Key

1) $l=\dfrac{\left(\sqrt{40}\right)^3-\left(\sqrt{13}\right)^3}{27}=7.63$

2) $2\dfrac{1}{14}$

3) $\dfrac{1097}{480}$

4) 21

5) $2\sqrt{3}$

6) 1.22

7) 2.32

8) 9

9) $\dfrac{14}{3}$

10) $\dfrac{123}{32}$

11) $\dfrac{\pi^2}{8}$

12) 12

13) 16

Work

Example Questions

1) A spring has a natural length of 20 inches. A 4 lbs force, starch it to 30 inches. How much work to stretch from 35 to 38 inches?

2) A Chain of 2lbs/ft is attached to a bucket of coal of 800lb at bottom of mine shaft, 500ft. Determine the amount of work needed to lift the bucket up the shaft.

3) Cylindrical tank is half full with oil of $\rho = 60\text{lb/ft}^2$. Its radius is 4ft and its height is 9ft. What is the work done in pumping out the oil to the top of the tank?

Center of mass

Example questions

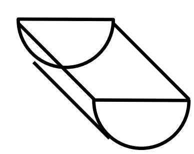
4) Find the center of mass of the shape created by the function $f(x) = 4 - x^2$ and the positive rays of the axis. Assume unit weight distribution.

5) Find the center of mass of the area between: $y = \sqrt{x}$ and $y = x^2$.

Hydrostatic Pressure and Force:

Example question

6) Given a tank of water which is half cylindrical as described. The radius of $r = 10\text{m}$. Find the hydrostatic force on the bases of the tank.



Mean Value Theorem for Integrals

Question:

7) Find the point(s) c, as in MTV for Integrals, for $f(x)=\frac{1}{x^2}$ on $[1,3]$.

Average Function Value

Question:

8) Find the average values if the following functions over the given intervals:

a. $g(x)=\frac{1}{x}$ on $[1,4]$.

b. $f(r)=\frac{3}{(1+r)^2}$ on $[1,6]$.

Numerical Integration

Questions:

9) Estimate $\int_0^2 e^{x^2}dx$ using each of the three rules (Midpoint, Trapezoid, Simpson) and $n=4$.

10) Estimate $\int_2^4 \frac{x}{x-1}dx$ using each of the three rules with its specified n.

a. Midpoint $(n=4)$

b. Trapezoid $(n=4)$

c. Simpson $(n=8)$

Answer Key:

1) 19.8 inch-lbs.
2) 650,000 ft-lbs.
3) $\cong 90,000$ ft-lbs.
4) $\left(\frac{3}{4}, \frac{8}{5}\right)$
5) $\left(\frac{9}{20}, \frac{9}{20}\right)$
6) 654,0000 Newtons
7) $c = \sqrt{3}$
8) a. $\frac{1}{3}\ln 4$ b. $\frac{3}{14}$
9) Midpoint: 14.48561253 Trapezoid: 20.64455905 Simpson: 17.35362645
10) a. 3.08975469 b. 3.11666667 c. 3.09872535