proprep™

# Workbook

# Complex Integration

## Questions

### Integral of Complex Function of Real Variable

**1)** Compute $\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx}e^{-imx}dx$ where $m,n \in \mathrm{Z}$ .

**2)** Compute the improper integral $\int_0^{\infty} e^{zt}dt$ where $z \in \square$ , $\operatorname{Re} z < 0$ .

### Contour Integral of a Complex Function

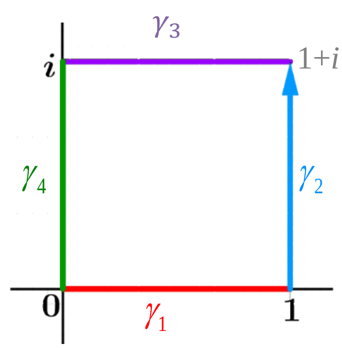
**3)** Evaluate. $\oint_{|z|=1} z^n dz \quad (n \in \square)$

**4)** Evaluate $\int_\gamma \frac{z+2}{z}dz$ where $\gamma:\ z(\theta) = 2e^{i\theta},\ 0 \le \theta \le \pi$

**5)** Evaluate $\int_\gamma (z-1)dz$ where $\gamma:\ z(\theta) = 1 + e^{i\theta},\ \pi \le \theta \le 2\pi$

**6)** Evaluate $\oint_\gamma \pi e^{\pi\bar{z}}dz$ on the square contour

$\gamma:\ 0 \rightarrow 1 \rightarrow 1+i \rightarrow i \rightarrow 0$ as illustrated.



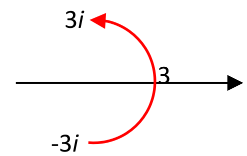

**7)** Compute the length of the segment $[z_1, z_2]$ where $z_1, z_2 \in \square$ .

**8)** Compute the length of the path $\gamma: z(t) = (t - \sin t) + i(1-\cos t),\ 0 \le t \le 1.$

**9)** Sketch the contour [curve] $\gamma: z(t) = ae^{it} + \frac{1}{ae^{it}},\ 0 \le t \le 2\pi$

## Integral Inequalities (Estimation Lemma)

**10)** Prove the inequality $\left|\int_\gamma \frac{z^3}{z^2+1}dz\right| \le \frac{81\pi}{3}$, where $\gamma:\begin{cases}|z|=3\\ \operatorname{Re}\{z\}\ge 0\end{cases}$




**11)** Prove the inequality $\left|\int_{|z|=3} \frac{1}{z^2-1}dz\right| \le \frac{3\pi}{4}$

**12)** Prove the inequality $\left|\int_C e^{z^2}dz\right| \le \sqrt{8}$ where $C$ is the line segment from $0$ to $2+2i$.

**13)** Prove the inequality $\left|\int_C \frac{z^2}{\sin z}dz\right| \le \frac{\pi^2}{2}+2$ where $C$ is the line segment from $\frac{\pi}{2}+i$ to $\frac{\pi}{2}-i$.

**14)** Let $a,b$ be two different complex numbers in the open left half-plane.

More precisely: $a,b\in\mathbb{C}$, $a\ne b,\ \operatorname{Re}\{a\}<0,\ \operatorname{Re}\{b\}<0$.

Prove that $|e^a-e^b|<|a-b|$.

## The Cauchy - Goursat Theorem

**15)** Compute the complex integral $\int_0^{2+\frac{i\pi}{4}} e^z dz$

**16)** Compute $\int_4^{1+i} \frac{1}{2\sqrt{z}}dz$, where we take the principal branch of $\sqrt{z}$.

## Cauchy's Integral Formula

**17)** Compute the complex integral $\oint\limits_{|z|=1} \frac{\cos z}{z} dz$

**18)** Compute the complex integral $\oint\limits_{|z-2|=1} \frac{e^z}{z-2} dz$

**19)** Compute the complex integral $\oint\limits_{|z-2|=1} \frac{\sin z^2}{z(z-2)} dz$

**20)** Compute the complex integral $\oint\limits_{|z|=1} \frac{e^z + e^{-z}}{z(z-2)(z-3)} dz$

**21)** Compute the complex integral $\oint\limits_{|z|=1.5} \frac{e^z}{z(z-1)(z-2)} dz$

**22)** Compute the complex integral $\oint\limits_{\gamma} \frac{\sin z}{z(z-2)(z-4)} dz$ where γ is as in the sketch.

**23)** Compute the complex integral $\oint\limits_{|z|=2} \frac{z^2 - e^{z^2}}{z(z^2-1)(z+3)} dz$

**24)** Prove the following:

Mean Value Property (of holomorphic functions)

Let *f* be holomorphic on a simply-connected domain *D* containing a circle *C*:

$C = \{z \mid |z-a| = r\}$ (geometrically) or $C = \{z = a + re^{i\theta} \mid 0 \le \theta \le 2\pi\}$ (parametrically).

Then $$f(a) = \frac{1}{2\pi} \int_0^{2\pi} f\left(a + r \cdot e^{i\theta}\right) d\theta$$ .

**25)** Compute the complex integral $\int_0^{\pi} \frac{1}{2+\sin 2\theta} d\theta$

**26)** Prove that $\int_0^{2\pi} \cos^{2n}(\theta)\, d\theta = \frac{2\pi}{2^{2n}} \binom{2n}{n}$ for any $n \in \mathbb{N}$. $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

**27)** Let $C$ be the curve $C=\left\{z\in\mathbb{C}\ \middle|\ |z|=2\ \wedge\ \mathrm{Im}\{z\}\geq 0\right\}$.

Compute $\int_C \frac{z}{z^2+1}dz$.

**28)** Compute the real integral $\int\limits_0^{2\pi}\frac{dx}{\left(a+b\cos x\right)}$, where $a>b>0$.

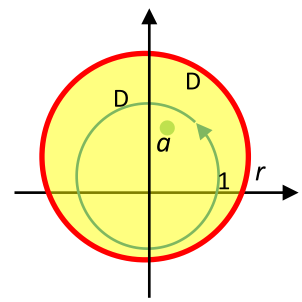
**29)** Let $f(z)$ be an entire function whose only zero is at $z=0$, and which satisfies $f'(0)=1$.

Compute $\oint_{|z|=1}\frac{\cos z}{f(z)}dz$.

**30)** Let $f(z)=u(z)+iv(z)$* be holomorphic in the domain $|z|<1$ and suppose that $u^2(0)=v^2(0)$. Prove that $\int\limits_0^{2\pi}u^2\left(re^{i\theta}\right)d\theta=\int\limits_0^{2\pi}v^2\left(re^{i\theta}\right)d\theta$ for all $0<r<1$.

**31)** Let $f$ be holomorphic in $D=\left\{z\in\mathbb{C}\ \middle|\ |z|<1\right\}$ and let $a\in\mathbb{C}$, $r\in R$ satisfy $0<|a|<r<1$. Prove that

$$\oint\limits_{|z|=r}\frac{\mathrm{Re}(z)}{z-a}f(z)dz=\pi i\left(\left[a+\frac{r^2}{a}\right]f(a)-\frac{r^2}{a}f(0)\right)$$




## Cauchy's Generalised Integral Formula

**32)** Use the "Feynman technique" to evaluate:

a) $\int_0^\infty x^2e^{-x}dx$

b) $\int_0^\infty x^3e^{-x}dx$

c) Can you guess, from the above, what is $\int_0^\infty x^ne^{-x}dx$ where *n* is natural?

**33)** Compute the contour integral $\oint\limits_{|z-i|=1}\frac{\sin z}{(z-i)^3}dz$

**34)** Compute the contour integral $\oint\limits_{|z|=1} \dfrac{\cos z}{z^3}\,dz$

**35)** Compute the contour integral $\oint\limits_{|z|=4} \dfrac{\cos z}{(z-\pi)^2}\,dz$

**36)** Compute the contour integral $\oint\limits_{|z-1|=1} \dfrac{\sin(\frac{\pi}{4}z)}{(z-1)^2(z-3)}\,dz$

**37)** Compute the contour integral $\oint\limits_{|z|=\frac{1}{2}} \dfrac{\sin\dfrac{\pi}{z+1}}{z^3}\,dz$

**38)** Compute the contour integral $\oint\limits_{|z|=4} \dfrac{\sin(\frac{\pi}{4}z)}{(z-1)^2(z-3)}\,dz$

**39)** Compute the contour integral $\oint\limits_{|z|=5} \dfrac{1}{(z-2)^2(z-4)}\,dz$

**40)** Use **Liouville's Theorem** to prove the following:

If $f$ is an entire function and $|f(z)| \le e^{\mathrm{Re}(z)}$ $\forall z \in \mathbb{C}$ then $f(z) = ce^z$ for some constant *c*.

## Primitive Functions

**41)**

a) Is the function $f(z) = \dfrac{1}{z^2+1}$ analytic in $\mathbb{C} - \{i, -i\}$ ? γ

b) Does it have a primitive there?

**42)**

a) Is the function $f(z) = \dfrac{1}{z^2+1}$ analytic in $\mathbb{C} - [-i, i]$ ?

b) Does it have a primitive there?

**43)**

Let $I = \int_{\Gamma} f(z)dz$ where $f(z) = \frac{2z}{z^2+1}$ and $\Gamma = \{z \in \mathbb{C} \mid |z| = 2, \operatorname{Im} z \geq 0\}$

a) Does $f(z)$ have a primitive in some simply-connected domain containing Γ?

b) Compute *I*.

**44)** Suppose that $f(z), g(z)$ are entire functions that satisfy $f^2(z) + g^2(z) = 1, \forall z \in \mathbb{C}$.

Prove that there exists an entire function $h(z)$ such that

$f(z) = \cos(h(z))$ and $g(z) = \sin(h(z))$

## Morera's Theorem

**45)** Let $\{f_n(z)\}$ be a sequence of analytic functions such that $\lim_{n\to\infty} f_n(z) \to f(z)$ uniformly on a simply-connected domain *D*. Prove that:

a) $f(z)$ is analytic on *D*.

b) $\forall z \in D \ \lim_{n\to\infty} f_n'(z) \to f'(z)$. [pointwise convergence, maybe not uniform].

## Uniform Convergence of a Sequence of Holomorphic Funcs

**46)** Let $a > 0$. Prove that the series $f(z) = \sum_{n=0}^{\infty} e^{-an^2 z}$ converges and is holomorphic on $\operatorname{Re}(z) > 0$.

**47)** Let $a > 0$. Prove that $f(z) = \sum_{n=0}^{\infty} \frac{1}{(a+n)^z}$ converges and is holomorphic on $\mathrm{Re}(z) > 1$.

[Recall that for real $x > 0$ we define $x^z = e^{z \cdot \ln(x)}$.]

**48)** Show that Riemann's zeta function $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$ is holomorphic on $\mathrm{Re}(z) > 1$.

Recall that we define $x^z = e^{z \cdot \ln(x)}$ for real $x > 0$.

**49)** The Laplace transform of a continuous and bounded function $f:(0,\infty) \to \mathrm{R}$ is the complex function $F(z) = \int_0^{\infty} f(x) e^{-zx} dx$ and we write $F = \mathrm{L}\{f\}$.

Prove that, if $F = \mathrm{L}\{f\}$ is continuous on $U = \{z \mid \mathrm{Re}(z) > 0\}$, it is holomorphic there.

## Summarizing Exercises

**Theorem:** Every bounded entire function must be constant. Prove it as follows:

**50)**

a) If $a \neq b \in \mathbb{C}$ then $\lim\limits_{R\to\infty} \oint\limits_{|z|=R} \frac{f(z)}{(z-a)(z-b)}dz = 0$

b) Show that $\frac{1}{(z-a)(z-b)} = \frac{1}{a-b}\left[\frac{1}{z-a} - \frac{1}{z-b}\right]$ (partial fractions)

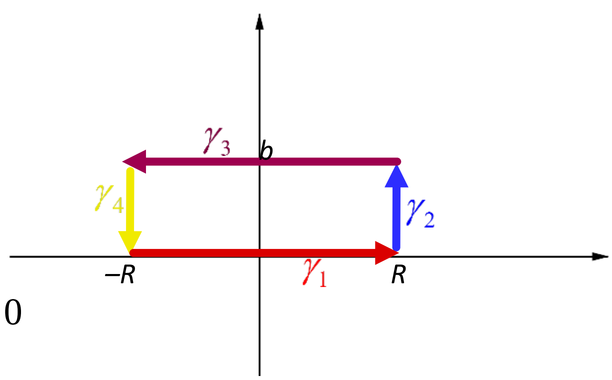
c) Use the above and Cauchy's Integral Formula to show that $f(a) = f(b)$

**51)**

Prove that, for $b > 0$, $\int\limits_{-\infty}^{\infty} e^{-x^2}\cos(2bx)dx = \sqrt{\pi}e^{-b^2}$. Proceed as follows:

Integrate the function $f(z) = e^{-z^2}$ on the path $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$ where

- $\gamma_1 = \{z = x \mid -R \leq x \leq R\}$
- $\gamma_2 = \{z = R + iy \mid 0 \leq y \leq b\}$
- $\gamma_3 = \{z = x + ib \mid x : R \to -R\}$
- $\gamma_4 = \{z = -R + iy \mid y : b \to 0\}$




and show that $\lim\limits_{R\to\infty} \int\limits_{\gamma_2} f(z)dz = \lim\limits_{R\to\infty} \int\limits_{\gamma_4} f(z)dz = 0$

**52)**

Prove that $\int\limits_0^{\infty} e^{-x^2}\cos(x^2)dx = \frac{\sqrt{\pi}}{2\sqrt[4]{2}}\cos\left(\frac{\pi}{8}\right)$ and $\int\limits_0^{\infty} e^{-x^2}\sin(x^2)dx = \frac{\sqrt{\pi}}{2\sqrt[4]{2}}\sin\left(\frac{\pi}{8}\right)$

Instruction: use the function $f(z) = e^{-\sqrt{2}z^2}$ and the contour $\gamma = \gamma_1 + C_R + \gamma_2$, where $\gamma_1, C_R, \gamma_2$ are as below, to show that $\lim\limits_{R\to\infty} \int\limits_{C_R} f(z)dz = 0$.

$\gamma_1 = \{z = x|0 \leq x \leq R\} C_R = \left\{z = Re^{i\theta} \mid 0 \leq \theta \leq \frac{\pi}{8}\right\} \gamma_2 = \left\{z = xe^{i\frac{\pi}{8}}|x: R \to 0\right\},$

## 53)

Let *f* be an entire function such that $f(z+1)=f(z)\ \forall z\in\mathbb{C}$

and suppose that $\underset{t\in[0,1]}{\overset{\max}{\sup}}\left|f(t+iR)\right|\xrightarrow{R\to\infty}0$. Prove that

$\int_0^1 Re\, f(t)\cdot Im\, f(t)dt=0$ and $\int_0^1 Re\, f^2(t)dt=\int_0^1 Im\, f^2(t)dt.$

## 54)

Prove that $\int\limits_{-\infty}^{\infty}\frac{\sin^2 x}{x^2}dx=\pi$. Note that $s(x)\equiv\frac{\sin^2 x}{x^2}$ is continuous if we define $s(0)=1$.

Proceed as follows:

a) Show that $s(x)=\mathrm{Re}\, f(x)$, where $f(z)=\frac{1-e^{2iz}}{2z^2}$. [*x* is real].

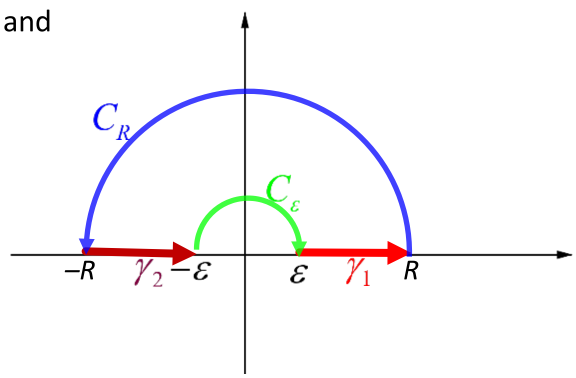
b) Explain why $\oint\limits_{\gamma} f(z)dz=0$, where $\gamma=\gamma_1+C_R+\gamma_2+C_\varepsilon$ and

$\gamma_1=\{z=x\,|\,\varepsilon\le x\le R\}$

$C_R=\{z=\mathrm{R}e^{i\theta}\,|\,0\le\theta\le\pi\}$

$\gamma_2=\{z=x\,|-R\le x\le-\varepsilon\}$

$C_\varepsilon=\{z=\varepsilon\mathrm{e}^{i\theta}\,|\,\theta:\pi\to 0\}$




c) Prove that $\lim\limits_{R\to\infty}\oint\limits_{C_R} f(z)dz=\mathbf{0}$ and $\lim\limits_{\varepsilon\to 0^+}\oint\limits_{C_\varepsilon} f(z)dz=\boldsymbol{\pi}$.

## 55)

Compute $\int\limits_{-\infty}^{\infty}\frac{\sin^6 x}{x^6}dx$. Note: $s(x)\equiv\frac{\sin^6 x}{x^6}$ is continuous if we define $s(0)=1$.

Hint: find $F(z)$ such that, for $x\in\mathrm{R}$, $s(x)=\mathrm{Re}\,F(x)$ or $s(x)=\mathrm{Im}\,F(x)$.

## Answer Key

**1)** $\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx}e^{-imx}dx = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$

**2)** $-\frac{1}{z}$

**3)** $\oint_{|z|=1} z^n dz = \begin{cases} 0 \;; & n \neq -1 \\ 2\pi i; & n = -1 \end{cases}$

**4)** $2\pi i - 4$

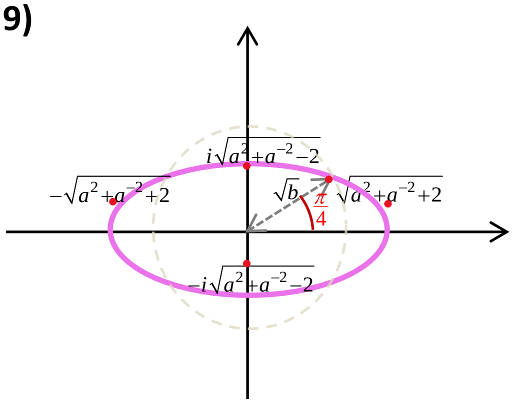
**5)** $0$

**6)** $4e^{\pi} - 4$

**7)** $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ , where $z_1 = x_1 + iy_1,\ z_2 = x_2 + iy_2$

**8)** $4\left[1 - \cos(0.5)\right] \approx 0.48$

**9)**




**10)** [Proof]

**11)** [Proof]

**12)** [Proof]

**13)** [Proof]

**14)** [Proof]

**15)** $\frac{e^2[1+i]}{\sqrt{2}} - 1$

**16)** $\left[\sqrt[4]{2}\, cos\frac{\pi}{8} - 2\right] + i\left[\sqrt[4]{2}\, sin\frac{\pi}{8}\right]$

**17)** $2\pi i$

**18)** $2\pi e^2 i$

**19)** $(\sin 4)\pi i$

**20)** $\frac{2\pi i}{3}$

**21)** $(1-2e)\pi$

**22)** $-\frac{\sin(2)\pi i}{2}$

**23)** $\pi i\left(\frac{2}{3} + \frac{3}{4}(1-e)\right)$

**24)** [proof]

**25)** $\frac{\pi}{\sqrt{3}}$

**26)** [proof]

**27)** $\pi i$

**28)** $\frac{2\pi}{\sqrt{a^2 - b^2}}$

**29)** $2\pi i$

**30)** [proof]

**31)** [proof]

**32)**

a) $\boxed{2}$

b) $\boxed{6}$

c) $\boxed{n!}$

**33)** $\boxed{\frac{\pi}{2}\left(e-\frac{1}{e}\right)}$

**34)** $\boxed{-\pi i}$

**35)** $\boxed{0}$

**36)** $\boxed{-\pi i\frac{\pi+2}{4\sqrt{2}}}$

**37)** $\boxed{-2\pi^2 i}$

**38)** $\boxed{\frac{-\pi^2 i}{4\sqrt{2}}}$

**39)** $\boxed{0}$

**40)** [proof]

**41)**

a) Yes

b) No

**42)**

a) Yes

b) Yes

**43)**

a) Yes

b) $2\pi i$

**44)** [proof]

**45)** [proof]

**46)** [proof]

**47)** [proof]

**48)** [proof]

**49)** [proof]

**50)** [proof]

**51)** [proof]

**52)** [proof]

**53)** [proof]

**54)** [proof]

**55)** $\frac{88\pi}{5 \cdot 2^5} = \boxed{\frac{11\pi}{20}}$