proprep™

Workbook

Complex Numbers

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Questions

1) Express in terms of i:

a. $\sqrt{-1} =$ b. $\sqrt{-4} =$ c. $\sqrt{-25} =$

d. $\sqrt{-3} =$ e. $\sqrt{-5} =$

2) Compute:

a. i^1 b. i^2 c. i^3

d. i^4 e. i^5 f. i^{17}

3) Write the values of a and b for the following complex numbers:

a. $2+5i$ b. $3-i$ c. $\frac{\sqrt{3}}{2}-\frac{1}{2}i$

d. $7i$ e. -4 f. 0

4) Solve the following quadratic equations:

a. $x^2=-1$ b. $x^2+36=0$ c. $x^2-2x+5=0$

5) Solve the following quadratic equation: $x^2+x+1=0$.

6) Solve the following quadratic equation: $z^2+iz+6=0$.

7) Given two complex numbers: $z_1=2+3i$, $z_2=5-2i$.
Compute the value of the following expressions:

a. z_1+z_2 b. z_1-z_2 c. $z_1\cdot z_2$

8) Write the complex conjugate of the following complex numbers:

a. $2+5i$ b. $3-i$ c. $\frac{\sqrt{3}}{2}-\frac{1}{2}i$

d. $7i$ e. -4 f. 0

9) Compute:

a. $\frac{11+2i}{2-i}$ b. $\frac{3+7i}{2-5i}$ c. $\frac{19-9i}{2-3i}$

10) Solve the following equation: $3z-11=iz-7i$.

11) Solve the following equation: $iz+5=4i$.

12) Solve the following system of equations in two complex unknowns, z and w: $\begin{cases} 3z+iw=5-4i \\ 5iz-2w=5+8i \end{cases}$.

13) Solve the following equations, in which a and b are real:

a. $2a-3i=10+bi$ b. $3a-8+5bi=2b-ai-3i$

14) Solve the following equation: $2z+7i=iz+\overline{z}-3$.

15) Compute the following square roots:

a. $\sqrt{5-12i}$ b. $\sqrt{8+6i}$

16) Solve the following equation: $z^2-2(1-2i)z-8i=0$.

17) Solve the following equation: $iz^2-2(1-i)z+6+15i=0$.

18) Solve the following equation: $z^2-i\overline{z}+6=0$.

19) Given the following equation in *z*: $(mi-2)z^2-2(m+2i)z+1=0$.

For which values of the complex parameter *m* does the equation:

a. have a single (unique) solution?
b. have no solution?

20) Write the following complex numbers in standard (rectangular) form:

a. $2(\cos 60°+i\sin 60°)$ b. $6(\cos 135°+i\sin 135°)$ c. $4(\cos 330°+i\sin 330°)$

d. $8(\cos(-30°)+i\sin(-30°))$ e. $4(\cos 690°+i\sin 690°)$ f. $8(\cos 90°+i\sin 90°)$

g. $3(\cos 270°+i\sin 270°)$ h. $\cos 180°+i\sin 180°$ i. $\cos 0°+i\sin 0°$

21) Write the following complex numbers in polar form:

a. $1+i$ b. $\sqrt{3}-i$ c. $-\frac{1}{2}-\frac{\sqrt{3}}{2}i$

d. $3+4i$ e. $6i$ f. $-i$

g. 4 h. -1 i. 1

j. 0

22) Compute the following expressions in polar form:

a. $2(\cos 120°+i\sin 120°)\cdot 3(\cos 60°+i\sin 60°)$ b. $(\cos 210°+i\sin 210°)\cdot 5(\cos(-40°)+i\sin(-40°))$

c. $\frac{12(\cos 315°+i\sin 315°)}{3(\cos 90°+i\sin 90°)}$ d. $\frac{1}{2(\cos 40°+i\sin 40°)}$

e. $6(\cos 30°+i\sin 30°)+2(\cos 210°+i\sin 210°)$

23) Given the complex number $z=r(\cos\theta+i\sin\theta)$.

Express the following in terms of r and θ:

a. $\overline{z}$ b. $1/z$ c. $-z$

d. $-\frac{1}{z}$ e. iz f. $z\cdot\overline{z}$

24) Let *z* be any complex number. Show that the following are purely real numbers:

a. $z+\overline{z}$ b. $z\cdot\overline{z}$ c. $\frac{z}{\overline{z}}+\frac{\overline{z}}{z}$

25) Let *z* be any complex number. Show that the following are purely imaginary numbers:

a. $z^2-\overline{z}^2$ b. $\frac{1}{\overline{z}}-\frac{1}{z}$

26) Prove the following identities:

a. $z-i\overline{z}=\overline{\overline{z}+iz}$ b. $z\cdot\overline{z}=|z|^2$

27) A square, whose sides are parallel to the axes, is inscribed in the canonical circle of radius $\sqrt{2}$ in the complex plane. Find its vertices.

28) A square, is inscribed in a canonical circle in the complex plane. Find its vertices, given that one of them is $1+\sqrt{3}i$.

29) An equilateral triangle, is inscribed in a canonical circle in the complex plane. Find its vertices, given that one of them is $1+\sqrt{3}i$.

30) An isosceles triangle, whose base angle is $30°$ is inscribed in a canonical circle in the complex plane. The apex is at $1+\sqrt{3}i$. Find the other vertices.

31) A complex number w is located outside of the unit circle in the complex plane. Decide if each of the following numbers is inside, outside or on the unit circle:

a. $\overline{w}$ b. $1/w$ c. $w/\overline{w}$ d. $w\cdot\overline{w}$

32) Use De Moivre's formula to evaluate the following expressions:

a. $\left[2\left(\cos 30°+i\sin 30°\right)\right]^3$ b. $\left[2\left(\cos 14°+i\sin 14°\right)\right]^5$ c. $\left(1+i\right)^4$

d. $\left(\sqrt{3}-i\right)^3$ e. $\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^{12}$

33) Solve the following equations:

a. $z^2=36\left(\cos 120°+i\sin 120°\right)$ b. $z^4=\left(9\left(\cos 80°+i\sin 80°\right)\right)^2$ c. $z^5=\frac{1}{2}+\frac{\sqrt{3}}{2}i$

34) Find the sum and the product of the 4th roots of unity.

35) Find the locus of points *z* in the complex plane such that $|z| = 2$.

36) Find the locus of points *z* in the complex plane such that $|z - 3i| = 5$.

37) Find the locus of points *z* in the complex plane such that $|z + i| + |\overline{z} + i| = |1 + 3i|$.

38) In a complex arithmetic sequence, the 3rd member is $a_3 = 5 - 9i$ and the 7th member is $a_7 = 13 + 3i$.
Find the sum of the first 10 members of the sequence.

39) In a complex geometric sequence, the 2nd member is $a_2 = 2 - 4i$ and the 5th member is $a_5 = 32 + 16i$.

a. Find the first member a_1 and the quotient q, given that q is purely imaginary.
b. Find the sum of the first 5 members of the sequence.

40) Let (z_1, z_2, z_3) be a geometric sequence.
Given $z_1 = 2$ and given that if we add $4i$ to z_3 we get an arithmetic sequence,
find the members of geometric sequence. [There are two possible answers.]

41) Solve the equation:
$z - \overline{z} + |z| = |2 - i|^2 - 4i + \mathrm{Im}(z) \cdot i$

42) Solve the equation:
$|2 - 3^{x^2 - x - 1} i| = \sqrt{13}$

43) Solve the equation $z^3 = \overline{z}$ in two ways:

a. Using rectangular form
b. Using polar form

44) **Prove**: If $az^2 + bz + c = 0 \ (a \neq 0)$ is a quadratic equation with real coefficients,
and if the equation has no real roots, then its roots are a pair of complex conjugates.

45) **Prove**: If z_1, z_2 are complex numbers which aren't purely real, and if $z_1 + z_2$ and $z_1 \cdot z_2$
are both real, then z_1, z_2 are conjugates.

46) **Prove**: If z is a complex number which isn't purely real, and if $z - \frac{1}{\overline{z}}$
is purely real, then z is on the unit circle.

47) **Prove** the formula for multiplication in polar form:
$r_1(\cos\theta_1 + i\sin\theta_1)\cdot r_2(\cos\theta_2 + i\sin\theta_2) = r_1 r_2\left(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right)$

48) Given a complex number *z* on the unit circle in the first quadrant,
and given that $\left|z^4 - z^3\right| = \sqrt{2-\sqrt{3}}$, find $\arg(z)$.

49) Given a complex number *z* on the unit circle. Find the value of the expression $z + iz$, given that it's purely real. Solve twice:

a. Using rectangular form
b. Using polar form

50) Given that z_1 and z_2 are the solutions of the quadratic $z^2 - 2\cos\theta\cdot z + 1 = 0$, and let P_1, P_2 be their corresponding points in the complex plane.
Find the size of the angle $\angle P_1OP_2$.

Answer Key

1) a. i b. $2i$ c. $5i$ d. $\sqrt{3i}$ e. $\sqrt{5i}$

2) a. i b. -1 c. $-i$ d. 1 e. i f. i

3) a. $a=2,\quad b=5$ b. $a=3,\quad b=-1$ c. $a=\frac{\sqrt{3}}{2},\quad b=-\frac{1}{2}$

d. $a=0,\quad b=7$ e. $a=-4,\quad b=0$ f. $a=0,\quad b=0$

4) a. $x=\pm i$ b. $x=\pm 6i$ c. $x=1\pm 2i$

5) $x=z=-\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$

6) $z_1=2i,\ z_2=-3i$

7) a. $7+i$ b. $-3+5i$ c. $16+11i$

8) a. $2-5i$ b. $3+i$ c. $\frac{\sqrt{3}}{2}+\frac{1}{2}i$ d. $-7i$ e. -4 f. 0

9) a. $4+3i$ b. $-1+i$ c. $5+3i$

10) $4-i$

11) $-4+5i$

12) $z=2-3i,\qquad w=5+i$

13) a. $a=5,\quad b=-3$ b. $a=2,\quad b=-1$

14) $z=-\frac{1}{2}-\frac{5}{2}i$

15) a. $\pm(3-2i)$, b. $\pm(3+i)$

16) $2,\ -4i$

17) $3i,\ -2-5i$

18) $2i,\ -3i$

19) a. m=I , b. m=-2i

20)

a. $1+\sqrt{3}i$ b. $-3\sqrt{2}+3\sqrt{2}i$ c. $2\sqrt{3}-2i$

d. $4\sqrt{3}-4i$ e. $2\sqrt{3}-2i$ f. $8i$

g. $-3i$ h. -1 i. 1

21)

a. $\sqrt{2}\left(\cos 45°+i\sin 45°\right)$ b. $2\left(\cos 270°+i\sin 270°\right)$ c. $\cos 240°+i\sin 240°$

d. $5\left(\cos 53.13°+i\sin 53.13°\right)$ e. $6\left(\cos 90°+i\sin 90°\right)$ f. $\cos 270°+i\sin 270°$

g. $4\left(\cos 0°+i\sin 0°\right)$ h. $\cos 180°+i\sin 180°$ i. $0\left(\cos 0°+i\sin 0°\right)$

j. 0

22)

a. $6\left(\cos 180°+i\sin 180°\right)$ b. $5\left(\cos 170°+i\sin 170°\right)$

c. $4\left(\cos 225°+i\sin 225°\right)$ d. $\frac{1}{2}\left(\cos(-40°)+i\sin(-40°)\right)$

e. $4\left(\cos 30°+i\sin 30°\right)$

23)

a. $r(\cos(-\theta))+i\sin(-\theta))$ or $r(\cos\theta-i\sin\theta)$

b. $\frac{1}{r}(\cos(-\theta))+i\sin(-\theta))$ or $\frac{1}{r}(\cos\theta-i\sin\theta)$

c. $-r(\cos(\theta))+i\sin(\theta))$ or $r\left(\cos(\theta+180°)+i\sin(\theta+180°)\right)$

d. $\frac{1}{r}(\cos(180°-\theta))+i\sin(180°-\theta))$ or $\frac{1}{r}(-\cos\theta+i\sin\theta)$

e. $r\left(\cos(\theta+90°)+i\sin(\theta+90°)\right)$ f. r^2 or $r^2\left(\cos 0°+i\sin 0°\right)$

24)

a. $2a$ b. a^2+b^2 c. $2(a^2-b^2)$

25)

a. $4abi$ b. $\frac{2b}{a^2+b^2}i$

26)

(proof)

27)

$1+i,\ -1+i,\ -1-i,\ 1-i$

28)

$1+\sqrt{3}i,\ -\sqrt{3}+i,\ -1-\sqrt{3}i,\ \sqrt{3}-i$

29)

$1+\sqrt{3}i,\ 2,\ 1-\sqrt{3}i$

30)

$1+\sqrt{3}i,\ -1+\sqrt{3}i,\ 2$

31)

a. outside b. inside c. on d. outside

32)

a. $8i$ b. $32\left(\cos 70° + i\sin 70°\right)$ c. -4

d. $-8i$ e. 1

33)

a. $z = 6\left(\cos 60° + i\sin 60°\right)$ or $6\left(\cos 240° + i\sin 240°\right)$

b. $z = 3\left(\cos 40° + i\sin 40°\right)$ or $3\left(\cos 130° + i\sin 130°\right)$ or $3\left(\cos 220° + i\sin 220°\right)$ or $3\left(\cos 310° + i\sin 310°\right)$

c. $z = \cos 12° + i\sin 12°$ or $\cos 84° + i\sin 84°$ or $\cos 156° + i\sin 156°$ or $\cos 228° + i\sin 228°$ or $\cos 300° + i\sin 300°$

34)

sum: 0

product: -1

35)

$x^2 + y^2 = 4 = 2^2$

Canonical circle with radius 2.

36)

$x^2 + (y-3)^2 = 25 = 5^2$

Circle centred at (0,3) with radius 5.

37)

$10x^2 + 6y^2 = 15$ (ellipse)

38)

$100 - 15i$

39)

a. $a_1 = 2 + i,\ q = -2i$
b. $20 + 25i$

40)

$(2,\ 4-2i,\ 3-4i)$ or $(2,\ 2i,\ -2)$

41)

$z = \pm 3 - 4i$

42)

$x = 2$ or $x = -1$

43)

$z = 0$ or 1 or i or -1 or $-i$

44)

(proof)

45)

(proof)

46)

(proof)

47)
(proof)

48)
$\arg(z) = 30^\circ$

49)
$\pm\sqrt{2}$

50)
$\angle P_1OP_2 = 2\theta$