Workbook

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Definite Integrals

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Questions

Compute the following integrals:

1) $\int_{1}^{4}\left(2x^{2}-4x+1\right)dx$

2) $\int_{0}^{2}\frac{2x+1}{x^{2}+x+1}dx$

3) $\int_{2}^{3}xe^{-x}dx$

4) $\int_{1}^{4}\frac{(\ln x)^{4}}{x}dx$

5) $\int_{1}^{\pi}\cos^{2}4xdx$

6) $\int_{0}^{4}f(x)\ ;\ f(x)=\begin{cases}\sqrt{x} & 0\le x\le 1\\ \frac{1}{x^{2}} & x\ge 1\end{cases}$

7) $\int_{-1}^{4}\sqrt{4+|x-1|}dx$

8) $\int_{0}^{\pi}\frac{x\sin x}{1+\cos^{2}x}dx$

9) $\int_{0}^{\pi/2}\frac{\sqrt[4]{\sin x}}{\sqrt[4]{\sin x}+\sqrt[4]{\cos x}}dx$

10) Let f be a continuous function. Prove that:

a. If f is even then $\int_{-a}^{a}f(x)dx=2\int_{0}^{a}f(x)dx$.

b. If f is odd then $\int_{-a}^{a}f(x)dx=0$.

11) Compute the following integrals:

a. $\int_{-4}^{4}\frac{\cos x}{x^{3}+x^{5}}dx$.

b. $\int_{-1}^{1}\frac{\sin x+1}{x^{2}+1}$.

Answer Key:

1) 13

2) $\ln 7$

3) $-4e^{-3}+3e^{-2}$

4) $\frac{1}{5}\ln^{5}4$

5) $\frac{1}{2}\left(\pi-1-\frac{1}{8}\sin 8\right)=1.062$

6) $1\frac{5}{12}$

7) $\frac{2}{3}\left(-16+6^{1.5}+7^{1.5}\right)=11.478$

8) $\frac{\pi^{2}}{4}$

9) $\frac{\pi}{4}=0.785$

10) Refer to the video

11) $\frac{\pi}{2}$

Inequalities

Questions:

1) Prove: $\frac{2}{41} \le \int_{-1}^{3} \frac{dx}{1+x} \le 4$.

2) Prove: $6 \le \int_{-4}^{2} \sqrt{1+x^2}\,dx \le 6\sqrt{17}$.

3) Prove: $2 \le \int_{0}^{2} e^{x^2}\,dx \le 2e^4$.

4) Prove: $\frac{1}{2}e^{-10} < \int_{0}^{10} \frac{e^{-x}}{x+10}\,dx < 1$.

5) Prove: $\frac{\pi}{14} < \int_{0}^{\pi/2} \frac{dx}{3+4\sin^2 x}\,dx \le \frac{\pi}{6}$.

6) Compute the integral: $-\frac{1}{2} \le \int_{0}^{1} x\sin\left(\frac{\ln x}{x+1}\right)dx \le \frac{1}{2}$.

7) Prove: $\int_{0}^{\pi} x^2 \arctan\left(\frac{\sin x}{x+4}\right)dx \le \frac{\pi^4}{6}$.

*Proof questions- for the solution see the videos.

Riemann Sum

Questions

1) Evaluate: $\lim_{n\to\infty} \frac{1^4+2^4+3^4+\cdots+n^4}{n^5}$

2) Evaluate: $\lim_{n\to\infty} \left\{ \frac{1}{n+1} + \frac{1}{n+2} + \cdots \frac{1}{n+n} \right\}$

3) Evaluate: $\lim_{n\to\infty} \left\{ \frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \cdots + \frac{n}{n^2+n^2} \right\}$

4) Evaluate: $\lim_{n\to\infty} \left\{ \frac{1}{\sqrt{n^2+1^2}} + \frac{1}{\sqrt{n^2+2^2}} + \cdots + \frac{1}{\sqrt{n^2+n^2}} \right\}$

5) Evaluate: $\lim_{n\to\infty} \left\{ \frac{\sqrt{n+1}+\sqrt{n+2}+\cdots\sqrt{2n}}{n^{3/2}} \right\}$

6) Evaluate: $\lim_{n\to\infty} \frac{\sin\frac{1}{n}+\sin\frac{2}{n}+\cdots+\sin\frac{n}{n}}{n}$

7) Evaluate: $\lim_{n\to\infty} \frac{1+\sqrt[n]{e}+\sqrt[n]{e^2}+\sqrt[n]{e^3}+\cdots\sqrt[n]{e^{n-1}}}{n}$

8) Evaluate: $\lim_{n\to\infty} \sum_{k=1}^{n} \ln \sqrt[n]{1+\frac{k}{n}}$

9) Use the Riemann sum definition of integral to evaluate $\int_0^x x dx$

Hint: $1+2+3+\cdots+n=0.5n(n+1)$.

10) Use the Riemann sum definition of integral to evaluate $\int_0^1 x^2 dx$

Hint: $1^2+2^2+3^2+\cdots+n^2=\frac{1}{6}n(n+1)(2n+1)$

11) Use the Riemann sum definition of integral to evaluate $\int_0^1 x^3 dx$

Hint: $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{1}{4}n^2(n+1)^2$

12) Use the Riemann sum definition of integral to evaluate $\int_0^\pi \sin x dx$

Hint: $\sin(\alpha) + \sin(2\alpha) + \cdots + \sin(n\alpha) = \frac{\sin\left(\frac{n}{2}\alpha\right)\sin\left(\frac{n+1}{2}\alpha\right)}{\sin\left(\frac{\alpha}{2}\right)}$

13) Use the Riemann sum definition of integral to evaluate $\int_2^5 (2x^2 + 3x) dx$

Hint: $1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n+1)$, $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$

Answer Key

1) $\frac{1}{5}$

2) $\ln 2$

3) $\frac{\pi}{4}$

4) $\ln\left(1+\sqrt{2}\right)$

5) 1.219

6) $1-\cos(1)$

7) $1-\cos(1)$

8) $2\ln 2-1$

9) $\frac{1}{2}$

10) $\frac{1}{3}$

11) $\frac{1}{4}$

12) 2

13) 109.5

Fundamental Theorem of Calculus

Questions

1) Recall the Fundamental Theorem of Calculus: $I(x)=\int_a^x f(t)dt \Rightarrow I'(x)=f(x)$.

Prove the following generalization: $I(x)=\int_a^{b(x)} f(t)dt \Rightarrow I'(x)=f(b(x))b'(x)$.

2) We previously generalized the FTC to: $I(x)=\int_a^{b(x)} f(t)dt \Rightarrow I'(x)=f(b(x))b'(x)$

Generalize again to prove: $I(x)=\int_{a(x)}^{b(x)} f(t)dt \Rightarrow I'(x)=f(b(x))b'(x)-f(a(x))a'(x)$.

3) Differentiate the following functions:

a. $I(x)=\int_2^x e^{-t^2}dt$

b. $I(x)=\int_1^{x^3} \frac{\ln t}{t^2}dt$

c. $I(x)=\int_2^{x^3+x} t\ln t dt$

d. $I(x)=\int_{x^3}^{x^2} \frac{dt}{\sqrt{1+t^4}}$

4) Evaluate the following limits:

a. $\lim_{x\to 0}\frac{\int_0^x \frac{tdt}{\cos t}}{\sin^2 x}$

b. $\lim_{x\to 0}\frac{1}{x^3}\int_0^{x^2}\sin\sqrt{t}dt$

c. $\lim_{x\to 4}\frac{x}{x-4}\int_4^x e^{t^2}dt$

5) Investigate the function $F(x)=\int_0^x (t+1)^4(t-1)^{10}dt$, particularly the following features:

a. Domain of definition
b. Extrema and intervals of increase/decrease
c. Inflection points and intervals of concavity/convexity

Answer Key

1) Proov.

2) Proov.

3) a. $I'(x)=e^{-x^2}$ b. $\frac{9\ln x}{x^4}$ c. $I'(x)=(x^3+x)\ln(x^3+x)(3x^2+1)$

d. $I'(x)=\frac{2x}{\sqrt{1+x^8}}-\frac{3x^2}{\sqrt{1+x^{12}}}$

4) a. $\frac{1}{2}$ b. $\frac{2}{3}$ c. $4e^{16}$

5) a. All x b. No extrema, the function is always increasing

c. Inflection points: -1, $-\frac{3}{7}$, 1

concave: $x<-1$ or $-\frac{3}{7}<x<1$

convex: $-1<x<-\frac{3}{7}$ or $x>1$