

# Workbook



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# Determinanat

## Determinanat

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### Questions

Evaluate the following determinants:

$$1) \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$2) \begin{vmatrix} 5 & 2 \\ -7 & 3 \end{vmatrix}$$

$$3) \begin{vmatrix} 4 & -1.5 \\ 2 & -1 \end{vmatrix}$$

$$4) \begin{vmatrix} 1 & 0 & 2 \\ 4 & 1 & 8 \\ 2 & 0 & 3 \end{vmatrix}$$

$$5) \begin{vmatrix} 2 & 1 & 1 \\ 0 & 2 & -1 \\ 1 & 0 & 0 \end{vmatrix}$$

$$6) \begin{vmatrix} 2 & 1 & 1 \\ 3 & -2 & 5 \\ 0 & 2 & 0 \end{vmatrix}$$

$$7) \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 3 & 0 \\ 1 & 2 & 3 & 4 \end{vmatrix}$$

$$8) \begin{vmatrix} 1 & 0 & 2 & 5 \\ -2 & 0 & -6 & 0 \\ 5 & 3 & -7 & 4 \\ 2 & 0 & 5 & 44 \end{vmatrix}$$

$$9) \begin{vmatrix} 4 & 0 & 0 & 5 \\ 1 & 7 & 2 & 4 \\ 4 & 0 & 0 & 0 \\ 1 & 4 & -1 & 1 \end{vmatrix}$$

$$10) \begin{vmatrix} 1 & 9 & 8 & 3 & 4 \\ 3 & 0 & -5 & 0 & 2 \\ 2 & -4 & 1 & 0 & 3 \\ 4 & 1 & 7 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \end{vmatrix}$$

$$11) \begin{vmatrix} 4 & 0 & 7 & 5 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ -7 & 2 & 1 & 5 & 9 \\ 3 & 0 & 4 & 2 & -1 \\ -5 & 0 & -8 & -3 & 2 \end{vmatrix}$$

### Answer key

- 1)  $ad - bc$
- 2) 29
- 3) -1
- 4) -1
- 5) -3
- 6) -14
- 7) 24
- 8) 234
- 9) -300
- 10) 9
- 11) 6

# Determinant

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## Rules of Determinants

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### Questions

1) Evaluate the following  $4 \times 4$  determinants [3 of them] by using row operations:

$$\text{a. } \begin{vmatrix} 1 & 3 & 0 & 2 \\ -2 & -5 & 7 & 4 \\ 3 & 5 & 2 & 1 \\ 2 & 2 & 2 & -1 \end{vmatrix}$$

$$\text{b. } \begin{vmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 5 & 4 & -3 \\ -1 & -2 & -1 & -1 \end{vmatrix}$$

$$\text{c. } \begin{vmatrix} 1 & -1 & -3 & 0 \\ 1 & 0 & 2 & 4 \\ -1 & 2 & 8 & 5 \\ 3 & -1 & -2 & 3 \end{vmatrix}$$

Evaluate the following  $5 \times 5$  determinants [3 of them] by using row operations:

$$\text{d. } \begin{vmatrix} 1 & 3 & 0 & 2 \\ -2 & -5 & 7 & 4 \\ 3 & 5 & 2 & 1 \\ 2 & 2 & 2 & -1 \end{vmatrix}$$

$$\text{e. } \begin{vmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 5 & 4 & -3 \\ -1 & -2 & -1 & -1 \end{vmatrix}$$

$$\text{f. } \begin{vmatrix} 1 & -1 & -3 & 0 \\ 1 & 0 & 2 & 4 \\ -1 & 2 & 8 & 5 \\ 3 & -1 & -2 & 3 \end{vmatrix}$$

Evaluate the following determinants:

$$\text{2) } \begin{vmatrix} 2 & 5 & -3 & -1 \\ 3 & 0 & 1 & -3 \\ -6 & 0 & -4 & 9 \\ 6 & 15 & -7 & -2 \end{vmatrix}$$

$$\text{3) } \begin{vmatrix} -1 & 2 & 3 & 0 \\ 3 & 4 & 3 & 0 \\ 5 & 4 & 6 & 6 \\ 3 & 4 & 7 & 3 \end{vmatrix}$$

$$\text{4) } \begin{vmatrix} 2 & 5 & 4 & 1 \\ 6 & 12 & 10 & 3 \\ 6 & -2 & -4 & 0 \\ -6 & 7 & 7 & 0 \end{vmatrix}$$

5) Prove (without computing) that the following three determinants are all zero:

$$\text{a. } \begin{vmatrix} 1 & 0 & 2 \\ 7 & 0 & 12 \\ 3 & 0 & 2 \end{vmatrix}$$

$$\text{b. } \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 5 & 7 & 9 \end{vmatrix}$$

$$\text{c. } \begin{vmatrix} 12 & 15 & 18 \\ 13 & 16 & 19 \\ 14 & 17 & 20 \end{vmatrix}$$

## Determinant

6) Prove (without computing) that the following three determinants are all zero:

$$\text{a. } \begin{vmatrix} y+z & z+x & y+x \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$$

$$\text{b. } \begin{vmatrix} a & a+x & a+y \\ b & b+x & b+y \\ c & c+x & c+y \end{vmatrix}$$

$$\text{c. } \begin{vmatrix} \sin^2 x & \cos^2 x & 1 \\ \sin^2 y & \cos^2 y & 1 \\ \sin^2 z & \cos^2 z & 1 \end{vmatrix}$$

$$\text{d. } \begin{vmatrix} 3 & -1 & 4 & 5 & 0 & 1 & -12 \\ -14 & 4 & 1 & -4 & 1 & 8 & 4 \\ 3 & 5 & -2 & 0 & -4 & 1 & -3 \\ -4 & 2 & 1 & 1 & 0 & 6 & -6 \\ -21 & 2 & 3 & 4 & 5 & 6 & 1 \\ 2 & -5 & 7 & -4 & 2.5 & -1 & -1.5 \\ -11 & 2 & -6 & 9 & -1 & 3 & 4 \end{vmatrix}$$

Given that:  $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 4$ , evaluate the following determinants:

$$\text{7) } \begin{vmatrix} a & g+d & 2d \\ b & h+e & 2e \\ c & i+f & 2f \end{vmatrix}$$

$$\text{8) } \begin{vmatrix} 2a-3d & 2d & g+4a \\ 2b-3e & 2e & h+4b \\ 2c-3f & 2f & i+4c \end{vmatrix}$$

$$\text{9) } \begin{vmatrix} 0 & g+3d & 3a & a+3d \\ 0 & h+3e & 3b & b+3e \\ 0 & i+3f & 3c & c+3f \\ 1 & 0 & 0 & 0 \end{vmatrix}$$

Evaluate the following determinants:

$$\text{10) } \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$\text{11) } \begin{vmatrix} 1 & x & x^2 & x^3 \\ 1 & y & y^2 & y^3 \\ 1 & z & z^2 & z^3 \\ 1 & t & t^2 & t^3 \end{vmatrix}$$

$$\text{12) } \det \begin{bmatrix} 1 & 1 & 1 & 1 & k \\ 1 & 1 & 1 & k & 1 \\ 1 & 1 & k & 1 & 1 \\ 1 & k & 1 & 1 & 1 \\ k & 1 & 1 & 1 & 1 \end{bmatrix}$$

## Determinanat

Evaluate the determinant  $\det(A)$ , where  $A = (a_{ij})$  is the  $n \times n$  matrix, given by:

$$13) a_{ij} = \begin{cases} 1 & i = j = 1 \\ 0 & i = j \neq 1 \\ j & i < j \\ -j & i > j \end{cases}$$

$$14) a_{ij} = \begin{cases} j & i = j + 1 \\ n & i = 1, j = n \\ 0 & \text{otherwise} \end{cases}$$

$$15) a_{ij} = \begin{cases} 1 & i + j = n + 1 \\ 0 & \text{otherwise} \end{cases}$$

$$16) a_{ij} = \begin{cases} a & i = j \\ b & \text{otherwise} \end{cases}$$

Evaluate the following determinants:

$$17) \det \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2 & \cdots & 2 \\ 1 & 2 & 3 & \cdots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & \cdots & n \end{bmatrix}$$

$$18) \det \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 3 & 3 & 3 & 3 & 3 & \cdots & 3 \\ 1 & 3 & 6 & 6 & 6 & 6 & \cdots & 6 \\ 1 & 3 & 6 & 9 & 9 & 9 & \cdots & 9 \\ 1 & 3 & 6 & 9 & 12 & 12 & \cdots & 12 \\ 1 & 3 & 6 & 9 & 12 & 15 & \cdots & 15 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 3 & 6 & 9 & 12 & 15 & \cdots & 3(n-1) \end{bmatrix}$$

19) Evaluate the determinant  $\det(A_{n \times n})$ , where  $A_{n \times n} = (a_{ij})$  is the  $n \times n$  matrix,

$$\text{given by: } a_{ij} = \begin{cases} a & i = j \\ b & i = j + 1 \\ c & j = i + 1 \\ 0 & \text{otherwise} \end{cases}$$

$$20) \text{ Evaluate the following: } \begin{vmatrix} a & b & c & d & e \\ f & g & h & i & j \\ k & l & m & n & o \\ p & q & r & s & t \\ 2a+1 & -2b & 1 & x & y \end{vmatrix} + \begin{vmatrix} a & b & c & d & e \\ f & g & h & i & j \\ k & l & m & n & o \\ p & q & r & s & t \\ -a-1 & 3b & c-1 & d-x & e-y \end{vmatrix}$$

### Answer Key

- 1) a. 0      b. 0      c. 3      d. 24      e. 44      f.104  
2) 120  
3) 114  
4) 6  
5) Proved as shown in the video.  
6) Proved as shown in the video.  
7) -8  
8) 16  
9) -36  
10)  $(b-a)(c-a)(c-b)$   
11)  $(y-x)(z-x)(t-x)(z-y)(t-y)(t-z)$   
12)  $(k-1)^4(k+4)$   
13)  $1 \cdot 2 \cdot 3 \cdot 4 \cdot \dots \cdot n = n!$   
14)  $(-1)^{n-1} n!$   
15)  $(-1)^{\frac{n(n+1)}{2}}$   
16)  $[a+(n-1)b](a-b)^{n-1}$   
17) 1  
18)  $2 \cdot 3^{n-2} \cdot 1$   
19)  $D_n = -1 + 2^{n+1}$   
20) 0



## More Rules of Determinants

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### Questions

1) Given that  $A$  and  $B$  are  $3 \times 3$  matrices satisfying  $|A|=4, |B|=2$ .

Evaluate the following determinants:

a.  $|ABA^{-1}B^T|$       b.  $|4A^2B^3|$       c.  $|-A^{-2}B^T A^3|$       d.  $|-2A^2A^T \text{adj}B|$

2) Given:  $(PQ)^{-1}APQ = B$ . Prove:  $|A|=|B|$ .

3) Let  $A$  and  $B$  be invertible matrices of order 4, such that  $2AB + 3I = 0$ ,  $|A|=2$ .  
Compute  $|B|$ .

4) Let  $A$  and  $B$  be invertible matrices of order 3, such that  $A + 3B = 0$ ,  $B^2 - 2A^{-1} = 0$ .  
Compute:  $|A|, |B|$ .

5) Prove:

a.  $|A^{-1}| = \frac{1}{|A|}$       b.  $|\text{adj}(A_{n \times n})| = |A|^{n-1}$

6) Let  $A$  be an antisymmetric matrix of odd order. Prove that  $|A|=0$ .

7) Given:  $A, B$  are matrices of order  $n$ ,  $B$  is invertible,  $|A|=128$ ,  $2AB = B^T A^2$ . Find  $n$ .

8) Given:  $\det(A_{n \times n}) = 2$ ,  $\det(B_{n \times n}) = \frac{1}{3}$ . Compute:  $\det\left(\frac{1}{3}B^{-n}A^{2n}\right)$ .

### Answer Key

- 1) a. 4      b.  $2^{13}$       c. -8      d.  $-2^{11}$
- 2) Proved as shown in the video.
- 3)  $|B| = \frac{81}{32}$
- 4)  $y = -\frac{2}{3} = |B|$  ,  $x = -27 \cdot \left(-\frac{2}{3}\right) = 18 = |A|$
- 5) Proved as shown in the video.
- 6) Proved as shown in the video.
- 7)  $n = 7$
- 8)  $4^n$

## Cramer's Rule

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### Questions

Solve, using Cramer's Rule, the systems:

$$1) \begin{cases} x + 2y = 5 \\ 3x + 4y = 11 \end{cases}$$

$$2) \begin{cases} x + z = 3 \\ 4x + y + 8z = 21 \\ 2x + 3z = 8 \end{cases}$$

$$3) \begin{cases} x + 2z + 5t = 8 \\ -2x - 6y = -8 \\ 5x + 3y - 7z + 4t = 5 \\ 2x + 5y + 44z = 51 \end{cases}$$

4) Consider the following system of equations:

$$\begin{aligned} kx + y + z + t + r &= 1 \\ x + ky + z + t + r &= 1 \\ x + y + kz + t + r &= 1 \\ x + y + z + kt + r &= 1 \\ x + y + z + t + kr &= 1 \end{aligned}$$

- For which values of  $k$  does the system have a unique solution?
- For which value of  $k$  does the system have a unique solution with  $x = \frac{1}{2}$ ?
- Is there a value of  $k$  such that the system has a unique solution with  $x = \frac{1}{5}$ ?
- Prove that if the system has a unique solution, then  $x = y = z = t = r$ .

### Answer Key

- 1)  $x = 1, y = 2$
- 2)  $x = 1, y = 1, z = 2$
- 3)  $x = 1, y = 1, z = 1, t = 1$
- 4) a. The system has a unique solution only if  $k \neq 1, k \neq -4$ 
  - b.  $k = -2$
  - c. No value of  $k$
  - d. Proved as shown in the video

## The Adjoint Matrix

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### Questions

1) Given the  $2 \times 2$  matrix  $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ . Compute  $\text{adj}(A)$  and use it to compute  $A^{-1}$ .

2) Given the  $3 \times 3$  matrix  $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & -1 \\ 5 & 2 & 3 \end{bmatrix}$ . Compute  $\text{adj}(A)$  and use it to compute  $A^{-1}$ .

3) Given the  $4 \times 4$  matrix  $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ . Compute  $\text{adj}(A)$  and use it to compute  $A^{-1}$ .

4) Given the  $5 \times 5$  matrix  $A = \begin{bmatrix} -9 & 26 & -1 & 14 & 10 \\ 13 & -7 & 87 & 4 & 0 \\ 71 & 35 & 3 & 0 & 0 \\ 17 & 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \end{bmatrix}$ .

Compute  $\text{adj}(A)_{1,5}$ , and use it to compute  $(A^{-1})_{1,5}$ .

5) Solve the following:

- Let  $A$  be a square matrix with  $\det(A) = 1$ . Prove that if all the elements of  $A$  are integers, then the elements of  $A^{-1}$  are integers too.
- Let  $A$  be an invertible lower-triangular matrix. Prove that  $A^{-1}$  is also lower-triangular.
- Prove that if  $A$  is invertible then so are  $\text{adj}(A)$  and  $A^T$ .
- If matrices  $A, B$  are invertible but  $C, D$  aren't, which of these is invertible:

i.  $C + D$

ii.  $A + B$

iii.  $AD$

iv.  $CD$

v.  $AB$

6) Find the values of  $k$  for which the following matrix is non-invertible:

$$\begin{bmatrix} 4 & 0 & 7 & 5 & 0 \\ 0 & 0 & 3k & 0 & 0 \\ -7k^2 & 2 & 4k & k & 9+k \\ 3 & 0 & 4 & 2 & -1 \\ -5 & 0 & -8 & -3 & 2 \end{bmatrix}$$

### Answer Key

1)  $\begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix}$

2) 1

3) -1

4)  $\frac{1}{2}$

5) a-c. For the solution see the video.      d.  $AB$

6) The matrix is non-invertible if and only if  $k = 0$ .

### Geometrical Applications of Determinants

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#### Questions

1) Answer the following:

a. Compute the area of the parallelograms whose vertices are given below:

1.  $(0,0), (5,2), (6,5), (11,6)$

2.  $(-1,0), (0,5), (1,-4), (2,1)$

b. Compute the volume of the tetrahedron with vertices:  $(0,0,0), (1,0,-2), (1,2,4), (7,1,0)$ .

c. Find the equation of the plane passing through the points:  $(3,3,-2), (-1,3,1), (1,1,-1)$ .

d. Compute the area of the triangle with vertices:  $(1,2), (3,4), (5,8)$ .



### Questions

- 1) a.1. 13      a.2. 14  
b. 22  
c.  $3x - y + 4z + 2 = 0$   
d.2