proprep™

Workbook

Elementary Functions

Elementary Functions

Questions

1) Solve the equation $e^z = -1$.

2) Solve the equation $\sin z = 2$.

3) Solve the equation $\cos z = 2$.

4) Consider the transformation $f(z) = z + 1$ and define the domain $U = \{z \in \mathbb{C} \mid |z| < 1\}$.

Find the image $f(U)$.

5) Consider the transformation $f(z) = 3z$ and define the domain $U = \{z \in \mathbb{C} \mid |z| < 1\}$.

Find the image $f(U)$.

6) Consider the transformation $f(z) = \frac{-1+i}{\sqrt{2}} \cdot z$ and define the domain

$U = \{z \in \mathbb{C} \mid \arg(z) = \frac{\pi}{4}\}$.

Find the image $f(U)$.

7) Find all possible values of $\sqrt{i}$.

8) Find all possible values of i^i.

9) Find all possible values of $2^{\frac{1}{9}+\frac{i}{50}}$.

10) Find any 3 values of $(2+2i)^{5i}$. How many are there altogether?

11) The following are 4 rules of exponents, where $c, d \in \mathbb{C}$; $z \in \mathbb{C}^{\times}$; $n \in \mathbb{Z}$

$1)\ z^{-c} = \frac{1}{z^c}$ $\qquad 2)\ z^{c+d} = z^c \cdot z^d$

$3)\ z^{c-d} = \frac{z^c}{z^d}$ $\qquad 4)\ (z^c)^n = z^{c \cdot n}$

Show that the 4th rule doesn't always hold if *n* is not an integer. Hint: compute $(i^2)^i$.

12) Given the function $F:\hat{\mathbb{C}} \to \mathbb{C}$, $F(z)=\operatorname{Log}\left(\frac{z-a}{z-b}\right)$, where $a,b \in \mathbb{C}$ and $a \neq b$.

Find a domain $U \subseteq \hat{\mathbb{C}}$, where *F* is analytic. [Assumes knowledge of Möbius transformations.]

13) Show that if $a>0$ then $\left|a^z\right| = a^{\operatorname{Re} z}$.

14) One way we could define the square root on $\mathbb{C}^\times$ is as follows:

if $z \neq 0$ then $z = re^{i\theta}$ for a unique $\theta \in [0, 2\pi)$, and then let $\sqrt{z} = \sqrt{r}e^{i\frac{\theta}{2}}$.

Note: this is not the principal value, where $\theta \in (-\pi, \pi]$.

Prove that this definition of $\sqrt{z}$ is not continuous.

15) Find all possible values of $\log(\log(-1))$.

16) Given: $\log(z)$ is a branch of $\log z$ on $\mathbb{C} - \{z = x+iy \mid x \geq 0, y = \sin x\}$ and $\log(1)=0$.

Compute the following values: $\log(-1)$, $\log i$, $\log(-i)$, $\log\frac{5\pi}{2}$, $\log\frac{3\pi}{2}$.

17) Let's define $\log_\alpha z = \ln|z| + i \cdot \arg z$ where we choose (uniquely) $\alpha - 2\pi < \arg z \leq \alpha$.

a) Compute the following values: $\log_0 1$, $\log_\pi 1$, $\log_{2\pi} 1$.

b) Find the image of $\mathbb{C} - (-\infty, 0]$ under the transformation $z \mapsto \log_\pi z$.

c) Find the image of $\mathbb{C} - [0, \infty)$ under the transformation $z \mapsto \log_0 z$.

18) Let $z_1, z_2, \ldots z_n \in \mathbb{C}$ be such that $\operatorname{Re} z_k > 0$ for $1 \leq k \leq n$ and $\operatorname{Re}(z_1 \cdot z_2 \cdot \ldots \cdot z_k) > 0$ for

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$1 \leq k \leq n-1$.

Prove: $\operatorname{Log}(z_1, z_2, \ldots z_n) = \operatorname{Log} z_1 + \ldots + \operatorname{Log} z_n$. [Log is the principal value of log.]

19) Let $\operatorname{Log} z$ be the principal branch of $\log z$ on $U = \mathbb{C} - (-\infty, 0]$.

Compute $\lim\limits_{y \to 0^+}\left[\operatorname{Log}(a+iy) - \operatorname{Log}(a-iy)\right]$ for $a>0$ and for $a<0$.

20) Let $D \subseteq \mathbb{C}$ be a domain containing 0. Prove that there's no analytic branch of $\sqrt[n]{z}$ in *D* $(n \geq 2)$.

That is, there doesn't exist an analytic *f* on *D* such that $f(z)^n = z$ for all $z \in D$.

21) Let $U \subseteq \mathbb{C}^{\times}$ be a domain (open and connected).

Let $f(z)$ and $g(z)$ be two continuous branches of $\log z$ on *U*.

Prove that $\exists k \in \mathbb{Z}$ such that $f - g = 2k\pi i$. I.e. $f(z) - g(z) = 2k\pi i \;\; \forall z \in U$.

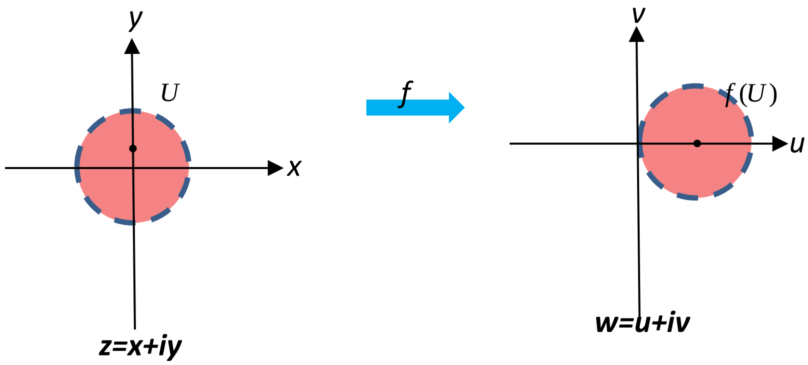
Answer Key

1) $z_n = (2n+1)\pi i$,Where n is any integer

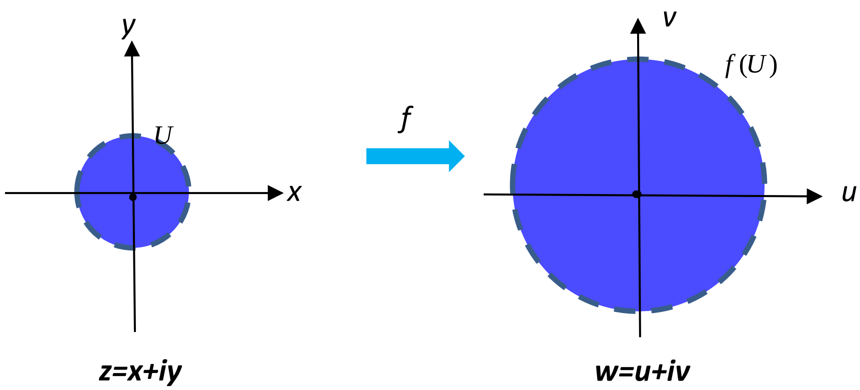
2) $z = \left(\frac{\pi}{2} + 2\pi n\right) + i \cdot \ln\left(2 \pm \sqrt{3}\right)$, where n is any integer.

3) $z = 2\pi n + i \cdot \ln\left(2 \pm \sqrt{3}\right)$, where n is any integer.

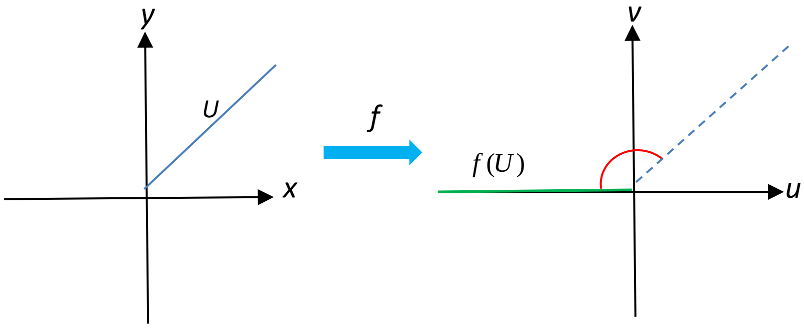
4) $f(U) = \{w \in \mathbb{C} \mid |w-1| < 1\}$ circle with radius 1 and center $(1,0)$



5) $f(U) = \{w \in \mathbb{C} \mid |w-1| < 1\}$ circle with radius 3 and center $(0,0)$



6) $f(U)=\{w\in\mathbb{C} \mid \arg(w)=\pi\}$



z=x+iy ***w=u+iv***

7) $z=\pm\left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)$

8) $\ldots, e^{\pi\frac{3}{2}}, e^{-\frac{1}{2}\pi}, e^{-\pi\frac{5}{2}}, e^{-\pi\frac{9}{2}}, \ldots$

9) $2^{\frac{1}{9}}e^{-\frac{2\pi k}{50}}e^{i\left(\frac{\ln(2)}{50}+\frac{2\pi k}{9}\right)}$ $\quad k\in\mathbb{Z}$

10) $e^{-5\left(\frac{\pi}{4}+2\pi k\right)}e^{i\cdot 5\ln\sqrt{8}}:\ \underbrace{e^{-5\left(\frac{\pi}{4}-2\pi\right)}e^{i\cdot 5\ln\left(\sqrt{8}\right)}}_{k=-1},\ \underbrace{e^{-5\cdot\frac{\pi}{4}}e^{i\cdot 5\ln\left(\sqrt{8}\right)}}_{k=0},\ \underbrace{e^{-5\left(\frac{\pi}{4}+2\pi\right)}e^{i\cdot 5\ln\left(\sqrt{8}\right)}}_{k=1}$

Infinitely many Answers.

11) (proof)

12) $U=\hat{\mathbb{C}}-[a,b]$

13) (proof)

14) (proof)

15)

$$\log\left(\log(-1)\right) = \begin{cases} \ln(\pi + 2\pi k) \ + i \cdot \left(\dfrac{\pi}{2} + 2\pi m\right) & k, m \in \mathbb{Z} \quad k \geq 0 \\ \ln(-\pi - 2\pi k) + i \cdot \left(-\dfrac{\pi}{2} + 2\pi m\right) & k, m \in \mathbb{Z} \quad k < 0 \end{cases} \quad \text{OR}$$

16)

$$\log(-1) = -\pi i$$

$$\log(i) = -\frac{3\pi}{2} i$$

$$\log(-i) = -\frac{\pi}{2} i$$

$$\log\frac{5\pi}{2} = \ln\frac{5\pi}{2}$$

$$\log\frac{3\pi}{2} = \ln\frac{3\pi}{2} - 2\pi i$$

17)

a) $\log_0 1 = 0$

$\log_\pi 1 = 0$

$\log_{2\pi} 1 = 2\pi i$

b) $\{w \mid -\pi < \operatorname{Im} w < \pi\}$

c) $\{w \mid -2\pi < \operatorname{Im} w < 0\}$

18) (proof)

19)

$a > 0 : \lim = 0$

$a < 0 : \lim = 2\pi i$

20) (proof)

21) (proof)