proprep

Workbook

Table of Contents

General Vector Spaces 2
Vector Spaces 2
Linear Combination, Dependence and Span 5
Vector Basis 8
Solution Space of Homogenous SLE 10
Subspaces 12
Coordinate Vectors and Change of Basis 14

General Vector Spaces

Vector Spaces

Questions

Check if W is a subspace of $M_n[\mathbb{R}]$, where:

1) $W=\{A|\ A=A^T\}$.

2) W is the set of matrices which *commute* with a given matrix B.
That is, $W=\{A|\ AB=BA\}$.

3) W is the set of matrices whose determinant is 0. That is, $W=\{A|\ \det A=0\}$.

4) W is the set of matrices which are equal to their own square.
That is, $W=\{A|\ A^2=A\}$.

5) W is the set of upper-triangular matrices.

6) W is the set of matrices whose product with a given matrix B is 0.
That is, $W=\{A|\ AB=\mathbf{0}\}$.

7) W is the set of matrices whose trace is 0.
That is, $W=\{A|\ tr(A)=0\}$.

8) W is the set of matrices such that the sum of each row is 0.

Check if W is a subspace of $P_n[\mathbb{R}]$, where:

9) W consists of the polynomials having 4 as a root. I.e., $W=\{p(x)|\ p(4)=0\}$.

10) W consists of the polynomials with degree ≤ 4. I.e., $W=\{p(x)|\ \deg(p)\leq 4\}$.

11) W consists of the polynomials with integer coefficients.

12) W consists of the polynomials with only even powers of x in its terms.

13) W consists of the polynomials having degree n where $4 \le n \le 7$.

14) $W = \left\{ p(x) \middle| p(0) = 1 \right\}$.

Check if W is a subspace of $F[\mathbb{R}]$, where:

15) W consists of all even functions. I.e., $W = \left\{ f(x) \mid f(x) = f(-x) \text{ for all } x \in \mathbb{R} \right\}$.

16) W consists of all bounded functions. I.e.,
$W = \left\{ f(x) \mid \left| f(x) \right| \le M \text{ for all } x \in \mathbb{R}, \text{ for some } M > 0 \right\}$.

17) W consists of all continuous functions.

18) W consists of all differentiable functions.

19) W consists of all constant functions.

20) $W = \left\{ f(x) \mid \int_0^1 f(x)\,dx = 4 \;(\text{assume } f \text{ is integrable}) \right\}$.

21) $W = \left\{ f(x) \mid f'(x) = 0 \;(\text{assume } f \text{ is differentiable}) \right\}$.

22) $W = \left\{ f(x) \mid f'(x) = 1 \;(\text{assume } f \text{ is differentiable}) \right\}$.

23) $W = \left\{ f \mid f(x+1) = f(x) \text{ for all } x \in \mathbb{R} \right\}$.

Check if W is a subspace of $\mathbb{C}^3[\mathbb{R}]$:

24) Check if W is a subspace of $\mathbb{C}^3[\mathbb{R}]$, where $W = \left\{ \langle z_1, z_2, z_3 \rangle \mid z_2 = \bar{z}_1, z_3 = z_1 + \bar{z}_1 \right\}$.

25) Check if $W = \left\{ \langle z_1, z_2, z_3 \rangle \mid z_2 = \bar{z}_1, z_3 = z_1 + \bar{z}_1 \right\}$ is a subspace of $\mathbb{C}^3$ (over the complex field $\mathbb{C}$).

Answer Key

1) Is a subspace
2) Is a subspace
3) Not a subspace
4) Not a subspace
5) Not a subspace
6) Not a subspace
7) Not a subspace
8) Not a subspace
9) Not a subspace
10) Not a subspace
11) Not a subspace
12) Is a subspace
13) Not a subspace
14) Not a subspace
15) Is a subspace
16) Is a subspace
17) Is a subspace
18) Is a subspace
19) Is a subspace
20) Not a subspace
21) Is a subspace
22) Not a subspace
23) Is a subspace
24) Is a subspace
25) Not a subspace

Linear Combination, Dependence and Span

Questions

1) We are given the following matrices in $M_2[\mathbb{R}]$:

$$A=\begin{bmatrix}4 & 1\\ 1 & 5\end{bmatrix},\ B=\begin{bmatrix}0 & 11\\ -5 & 3\end{bmatrix},\ C=\begin{bmatrix}2 & -5\\ 3 & 1\end{bmatrix},\ D=\begin{bmatrix}1 & 3\\ -1 & 2\end{bmatrix}$$

a. Are these matrices linearly dependent?
b. If so, try to write each of them as a linear combination of the rest.
c. Does A belong to $Sp\{B,C\}$?

2) We are given the following polynomials in $P_3[\mathbb{R}]$:

$$p_1(x)=4+x+x^2+5x^3\ ,\ p_2(x)=11x-5x^2+3x^3\ ,$$
$$p_3(x)=2-5x+3x^2+x^3\ ,\ p_4(x)=1+3x-x^2+2x^3$$

a. Are these polynomials linearly dependent?
b. If so, try to write each of them as a linear combination of the rest.
c. Does p_2 belong to $Sp\{p_1,p_3\}$?

3) We are given the following set of vectors in $\mathbb{R}^3$: $S=\{\langle c,2,4\rangle,\langle 2.4,a,2\rangle,\langle c,b,6\rangle,\langle b,2,a\rangle\}$
For which values of a,b,c is S linearly dependent?

4) We are given that the set $\{u,v,w\}$ of vectors is linearly independent in $V[F]$.

a. Is the set $\{u-v,u-w,u+v-2w\}$ linearly dependent?
b. If so, try to write each vector in the set as a linear combination of the others.

5) We are given that the set $\{u,v,w\}$ of vectors is linearly independent in $V[F]$.

a. Is the set $\{u+v,v+w,w\}$ linearly dependent?
b. If so, try to write vector in the set as a linear combination of the others.

6) We are given that the set $\{u,v,w\}$ of vectors is linearly independent in $V[F]$.

a. Is the set $\{u+2v+3w,4u+5v+6w,7u+8v+9w\}$ linearly dependent?
b. If so, try to write each vector in the set as a linear combination of the others.

7) Is the set of vectors $\left\{\langle 1,i,i-1\rangle,\langle i+1,i-1,-2\rangle\right\}$ linearly independent in $\mathbb{C}^3\left[\mathbb{C}\right]$?

8) Is the set of vectors $\left\{\langle 1,i,i-1\rangle,\langle i+1,i-1,-2\rangle\right\}$ linearly independent in $\mathbb{C}^3\left[\mathbb{R}\right]$?

Answer Key

1) a. Yes, they're linearly dependent.

b. $A = B + 2C$, $B = A - 2C$, $C = \frac{1}{2}A - \frac{1}{2}B$, $D = \frac{1}{4}A + \frac{1}{4}B$

c. Yes, follows from $A = B + 2C$.

2) a. Yes, they are linearly dependent.

b. $p_1 = p_2 + 2p_3$, $p_2 = p_1 - 2p_3$, $p_3 = \frac{1}{2}p_1 - \frac{1}{2}p_2$, $p_4 = \frac{1}{4}p_1 + \frac{1}{4}p_2$

c. Yes, follows from $p_2 = p_1 - 2p_3$.

3) For all values a, b, c S linearly is dependent.

4) a. Yes, they are linearly dependent.

b. $x = 2y - z$
$y = 0.5x + 0.5z$
$z = 2y - x$

5) a. No

b. N/A

6) a. Yes, they are linearly dependent.

b. $x = 2y - z$
$y = 0.5x + 0.5z$
$z = 2y - x$

7) No, the vectors are linearly dependent.

8) The vectors are linearly independent.

Vector Basis

Questions

1) Check if each of the following sets is a basis of $M_{2\times 2}[\mathbb{R}]$ (A.K.A $M_2[\mathbb{R}]$):

a. $\left\{\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix},\begin{bmatrix}5 & 6\\7 & 8\end{bmatrix},\begin{bmatrix}9 & 1\\2 & 3\end{bmatrix}\right\}$

b. $\left\{\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix},\begin{bmatrix}5 & 6\\7 & 8\end{bmatrix},\begin{bmatrix}9 & 1\\2 & 3\end{bmatrix},\begin{bmatrix}5 & 6\\7 & 2\end{bmatrix},\begin{bmatrix}5 & 16\\7 & 8\end{bmatrix}\right\}$

c. $\left\{\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix},\begin{bmatrix}0 & 1\\1 & 1\end{bmatrix},\begin{bmatrix}0 & 0\\1 & 1\end{bmatrix},\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right\}$

2) Check if each of the following sets is a basis of $P_2[\mathbb{R}]$ (deg ≤ 2 poly):

a. $\{1+x,\ x^2+2x+3\}$

b. $\{1+x,\ x^2+2x+3, 2x+4x^2, x-x^2\}$

c. $\{1+2x+3x^2,\ 4+5x+6x^2, 7+8x+10x^2\}$

Answer Key

1) a. No, the three vectors can't form a basis.

b. No, the five vectors can't form a basis.

c. Yes, the four vectors do form a basis.

2) a. No, the two vectors can't form a basis.

b. No, the four vectors can't form a basis.

c. Yes, the three vectors do form a basis.

Solution Space of Homogenous SLE

Questions

1) Let $U = \{A \in M_2[\mathbb{R}] \mid A = A^T\}$. Symmetric 2x2 matrices.

Find a basis and the dimension of U.

2) Let $U = \left\{A \in M_2[R] \,\middle|\, A \cdot \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right\}$.

Find a basis and the dimension of U.

3) Let $U = \{p(x) \in P_3[\mathbb{R}] \mid p(1) = 0\}$

Find a basis and the dimension of U.

Answer Key

1) $B_U = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}$, $\dim U = 3$.

2) $U = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$, $\dim U = 0$, $B_U = \varnothing$ [empty set].

3) $B_U = \{ p_1(x) = -1 + x^3 \;\; p_2(x) = -1 + x^2, p_3(x) = -1 + x \}$, $\dim U = 3$

Subspaces

Questions

1) Consider the subspace of $M_2[\mathbb{R}]$ defined as follows:

$$U = span\left\{\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} 4 & 1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & -1 \\ 1 & -3 \end{bmatrix}\right\}.$$

Find a basis and the dimension of U .

2) Consider the subspace of $P_3[\mathbb{R}]$ defined as follows:

$$U = span\{1+x-x^2+2x^3, 4+x-x^2+x^3, 2-x+x^2-3x^3\}$$

Find a basis and the dimension of U .

Answer Key

1) $B_U = \left\{ \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & -3 \\ 3 & -7 \end{bmatrix} \right\}, \ \dim U = 2$

2) $B_U = \left\{ 1 + x - x^2 + 2x^3, -3x + 3x^2 - 7x^3 \right\}, \ \dim U = 2$

Coordinate Vectors and Change of Basis

Questions

1) Given the following two bases of $P_2[\mathbb{R}]$:

$B_1 = \{1+x, x, x+x^2\}$; and $B_2 = \{1+x^2, x+x^2, x^2\}$, and let $p(x) = a+bx+cx^2$,

be a general polynomial in $P_2[\mathbb{R}]$.

Compute $[p(x)]_{B_1}$, the coordinate vector of $p(x)$ relative to B_1 and B_2.

Find the change-of-basis matrix from B_1 to B_2.

2) Given the following two bases of $M_2[\mathbb{R}]$:

$$B = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$E = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Compute the coordinate vector of $X = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$, relative to B and E.

Find the change-of-basis matrix from B to E.

Answer Key

1) $[v]_{B_1} = \langle a, b-a-c, c \rangle$; $[v]_{B_2} = \langle a, b, c-a-b \rangle$, $[M]_{B_1}^{B_2} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$

2) $[X]_B = \langle x, \mathrm{y} - \mathrm{x}, z-y+x, t-z+y-x \rangle$

E is the elementary or standard basis of $M_2[\mathbb{R}]$: $[X]_E = \langle x, \mathrm{y}, z, t \rangle$

The change-of-basis matrix from B to E: $[M]_B^E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 1 \end{bmatrix}$