proprep™

Workbook

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Determinants

Determinants

Questions

Evaluate the following determinants:

1) $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

2) $\begin{vmatrix} 5 & 2 \\ -7 & 3 \end{vmatrix}$

3) $\begin{vmatrix} 4 & -1.5 \\ 2 & -1 \end{vmatrix}$

4) $\begin{vmatrix} 1 & 0 & 2 \\ 4 & 1 & 8 \\ 2 & 0 & 3 \end{vmatrix}$

5) $\begin{vmatrix} 2 & 1 & 1 \\ 0 & 2 & -1 \\ 1 & 0 & 0 \end{vmatrix}$

6) $\begin{vmatrix} 2 & 1 & 1 \\ 3 & -2 & 5 \\ 0 & 2 & 0 \end{vmatrix}$

7) $\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 3 & 0 \\ 1 & 2 & 3 & 4 \end{vmatrix}$

8) $\begin{vmatrix} 1 & 0 & 2 & 5 \\ -2 & 0 & -6 & 0 \\ 5 & 3 & -7 & 4 \\ 2 & 0 & 5 & 44 \end{vmatrix}$

9) $\begin{vmatrix} 4 & 0 & 0 & 5 \\ 1 & 7 & 2 & 4 \\ 4 & 0 & 0 & 0 \\ 1 & 4 & -1 & 1 \end{vmatrix}$

10) $\begin{vmatrix} 1 & 9 & 8 & 3 & 4 \\ 3 & 0 & -5 & 0 & 2 \\ 2 & -4 & 1 & 0 & 3 \\ 4 & 1 & 7 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \end{vmatrix}$

11) $\begin{vmatrix} 4 & 0 & 7 & 5 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ -7 & 2 & 1 & 5 & 9 \\ 3 & 0 & 4 & 2 & -1 \\ -5 & 0 & -8 & -3 & 2 \end{vmatrix}$

Answer Key

1) $ad-bc$ **2)** 29 **3)** -1 **4)** -1
5) -3 **6)** -14 **7)** 24 **8)** 234
9) -300 **10)** 9 **11)** 6

Rules of Determinants

Questions

1) Evaluate the following 4×4 determinants [3 of them] by using row operations:

a. $\begin{vmatrix} 1 & 3 & 0 & 2 \\ -2 & -5 & 7 & 4 \\ 3 & 5 & 2 & 1 \\ 2 & 2 & 2 & -1 \end{vmatrix}$ b. $\begin{vmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 5 & 4 & -3 \\ -1 & -2 & -1 & -1 \end{vmatrix}$ c. $\begin{vmatrix} 1 & -1 & -3 & 0 \\ 1 & 0 & 2 & 4 \\ -1 & 2 & 8 & 5 \\ 3 & -1 & -2 & 3 \end{vmatrix}$

Evaluate the following 5×5 determinants [3 of them] by using row operations:

d. $\begin{vmatrix} 1 & 3 & 0 & 2 \\ -2 & -5 & 7 & 4 \\ 3 & 5 & 2 & 1 \\ 2 & 2 & 2 & -1 \end{vmatrix}$ e. $\begin{vmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 5 & 4 & -3 \\ -1 & -2 & -1 & -1 \end{vmatrix}$ f. $\begin{vmatrix} 1 & -1 & -3 & 0 \\ 1 & 0 & 2 & 4 \\ -1 & 2 & 8 & 5 \\ 3 & -1 & -2 & 3 \end{vmatrix}$

Evaluate the following determinants:

2) $\begin{vmatrix} 2 & 5 & -3 & -1 \\ 3 & 0 & 1 & -3 \\ -6 & 0 & -4 & 9 \\ 6 & 15 & -7 & -2 \end{vmatrix}$ **3)** $\begin{vmatrix} -1 & 2 & 3 & 0 \\ 3 & 4 & 3 & 0 \\ 5 & 4 & 6 & 6 \\ 3 & 4 & 7 & 3 \end{vmatrix}$ **4)** $\begin{vmatrix} 2 & 5 & 4 & 1 \\ 6 & 12 & 10 & 3 \\ 6 & -2 & -4 & 0 \\ -6 & 7 & 7 & 0 \end{vmatrix}$

5) Prove (without computing) that the following three determinants are all zero:

a. $\begin{vmatrix} 1 & 0 & 2 \\ 7 & 0 & 12 \\ 3 & 0 & 2 \end{vmatrix}$ b. $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 5 & 7 & 9 \end{vmatrix}$ c. $\begin{vmatrix} 12 & 15 & 18 \\ 13 & 16 & 19 \\ 14 & 17 & 20 \end{vmatrix}$

6) Prove (without computing) that the following three determinants are all zero:

a. $\begin{vmatrix} y+z & z+x & y+x \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$

b. $\begin{vmatrix} a & a+x & a+y \\ b & b+x & b+y \\ c & c+x & c+y \end{vmatrix}$

c. $\begin{vmatrix} \sin^2 x & \cos^2 x & 1 \\ \sin^2 y & \cos^2 y & 1 \\ \sin^2 z & \cos^2 z & 1 \end{vmatrix}$

d. $\begin{vmatrix} 3 & -1 & 4 & 5 & 0 & 1 & -12 \\ -14 & 4 & 1 & -4 & 1 & 8 & 4 \\ 3 & 5 & -2 & 0 & -4 & 1 & -3 \\ -4 & 2 & 1 & 1 & 0 & 6 & -6 \\ -21 & 2 & 3 & 4 & 5 & 6 & 1 \\ 2 & -5 & 7 & -4 & 2.5 & -1 & -1.5 \\ -11 & 2 & -6 & 9 & -1 & 3 & 4 \end{vmatrix}$

Given that: $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 4$, evaluate the following determinants:

7) $\begin{vmatrix} a & g+d & 2d \\ b & h+e & 2e \\ c & i+f & 2f \end{vmatrix}$

8) $\begin{vmatrix} 2a-3d & 2d & g+4a \\ 2b-3e & 2e & h+4b \\ 2c-3f & 2f & i+4c \end{vmatrix}$

9) $\begin{vmatrix} 0 & g+3d & 3a & a+3d \\ 0 & h+3e & 3b & b+3e \\ 0 & i+3f & 3c & c+3f \\ 1 & 0 & 0 & 0 \end{vmatrix}$

Evaluate the following determinants:

10) $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

11) $\begin{vmatrix} 1 & x & x^2 & x^3 \\ 1 & y & y^2 & y^3 \\ 1 & z & z^2 & z^3 \\ 1 & t & t^2 & t^3 \end{vmatrix}$

12) $\det \begin{bmatrix} 1 & 1 & 1 & 1 & k \\ 1 & 1 & 1 & k & 1 \\ 1 & 1 & k & 1 & 1 \\ 1 & k & 1 & 1 & 1 \\ k & 1 & 1 & 1 & 1 \end{bmatrix}$

Evaluate the determinant $\det(A)$, where $A=(a_{ij})$ is the $n\times n$ matrix, given by:

13) $a_{ij}=\begin{cases}1 & i=j=1\\ 0 & i=j\neq 1\\ j & i<j\\ -j & i>j\end{cases}$

14) $a_{ij}=\begin{cases}j & i=j+1\\ n & i=1,\ j=n\\ 0 & otherwise\end{cases}$

15) $a_{ij}=\begin{cases}1 & i+j=n+1\\ 0 & otherwise\end{cases}$

16) $a_{ij}=\begin{cases}a & i=j\\ b & otherwise\end{cases}$

Evaluate the following determinants:

17) $\det\begin{bmatrix}1 & 1 & 1 & \cdots & 1\\ 1 & 2 & 2 & \cdots & 2\\ 1 & 2 & 3 & \cdots & 3\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & 2 & 3 & \cdots & n\end{bmatrix}$

18) $\det\begin{bmatrix}1 & 1 & 1 & 1 & 1 & 1 & \cdots & 1\\ 1 & 3 & 3 & 3 & 3 & 3 & \cdots & 3\\ 1 & 3 & 6 & 6 & 6 & 6 & \cdots & 6\\ 1 & 3 & 6 & 9 & 9 & 9 & \cdots & 9\\ 1 & 3 & 6 & 9 & 12 & 12 & \cdots & 12\\ 1 & 3 & 6 & 9 & 12 & 15 & \cdots & 15\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & 3 & 6 & 9 & 12 & 15 & \cdots & 3(n-1)\end{bmatrix}$

19) Evaluate the determinant $\det(A_{n\times n})$, where $A_{n\times n}=(a_{ij})$ is the $n\times n$ matrix,

given by: $a_{ij}=\begin{cases}a & i=j\\ b & i=j+1\\ c & j=i+1\\ 0 & otherwise\end{cases}$

20) Evaluate the following: $\begin{vmatrix}a & b & c & d & e\\ f & g & h & i & j\\ k & l & m & n & o\\ p & q & r & s & t\\ 2a+1 & -2b & 1 & x & y\end{vmatrix}+\begin{vmatrix}a & b & c & d & e\\ f & g & h & i & j\\ k & l & m & n & o\\ p & q & r & s & t\\ -a-1 & 3b & c-1 & d-x & e-y\end{vmatrix}$

Answer Key

1) a. 0 b. 0 c. 3 d. 24 e. 44 f.104

2) 120

3) 114

4) 6

5) Proved as shown in the video.

6) Proved as shown in the video.

7) -8

8) 16

9) -36

10) $(b-a)(c-a)(c-b)$

11) $(y-x)(z-x)(t-x)(z-y)(t-y)(t-z)$

12) $(k-1)^4(k+4)$

13) $1\cdot 2\cdot 3\cdot 4\cdot \ldots \cdot n = n!$

14) $(-1)^{n-1}n!$

15) $(-1)^{\frac{n(n+1)}{2}}$

16) $\left[a+(n-1)b\right](a-b)^{n-1}$

17) 1

18) $2\cdot 3^{n-2}\cdot 1$

19) $D_n = -1+2^{n+1}$

20) 0

More Rules of Determinants

Questions

1) Given that A and B are 3×3 matrices satisfying $|A|=4, |B|=2$.
Evaluate the following determinants:

a. $\left|ABA^{-1}B^T\right|$ b. $\left|4A^2B^3\right|$ c. $\left|-A^{-2}B^TA^3\right|$ d. $\left|-2A^2A^T adjB\right|$

2) Given: $(PQ)^{-1}APQ=B$. Prove: $|A|=|B|$.

3) Let A and B be invertible matrices of order 4, such that $2AB+3I=0$, $|A|=2$.
Compute $|B|$.

4) Let A and B be invertible matrices of order 3, such that $A+3B=0$, $B^2-2A^{-1}=0$.
Compute: $|A|, |B|$.

5) Prove: a. $|A^{-1}|=\dfrac{1}{|A|}$ b. $\left|adj\left(A_{n\times n}\right)\right|=|A|^{n-1}$

6) Let A be an antisymmetric matrix of odd order. Prove that $|A|=0$.

7) Given: A, B are matrices of order n, B is invertible, $|A|=128$, $2AB=B^TA^2$. Find n.

8) Given: $\det\left(A_{n\times n}\right)=2, \det\left(B_{n\times n}\right)=\frac{1}{3}$. Compute: $\det\left(\dfrac{1}{3}B^{-n}A^{2n}\right)$.

Answer Key

1) a. 4 b. 2^{13} c. -8 d. -2^{11}

2) Proved as shown in the video.

3) $|B|=\dfrac{81}{32}$

4) $y=-\dfrac{2}{3}=|B|$, $x=-27\cdot\left(-\dfrac{2}{3}\right)=18=|A|$

5)-6) Proved as shown in the video.

7) $n=7$

8) 4^n

Cramer's rule

Questions

Solve, using Cramer's Rule, the systems:

1) $\begin{cases} x+2y=5 \\ 3x+4y=11 \end{cases}$

2) $\begin{cases} x+z=3 \\ 4x+y+8z=21 \\ 2x+3z \quad =8 \end{cases}$

3) $\begin{cases} x+2z+5t=8 \\ -2x-6y=-8 \\ 5x+3y-7z+4t=5 \\ 2x+5y+44z=51 \end{cases}$

4) Consider the following system of equations: $\begin{matrix} kx+y+z+t+r=1 \\ x+ky+z+t+r=1 \\ x+y+kz+t+r=1 \\ x+y+z+kt+r=1 \\ x+y+z+t+kr=1 \end{matrix}$.

a. For which values of k does the system have a unique solution?

b. For which value of k does the system have a unique solution with $x=\frac{1}{2}$?

c. Is there a value of k such that the system has a unique solution with $x=\frac{1}{5}$?

d. Prove that if the system has a unique solution, then $x=y=z=t=r$.

Answer Key

1) $x=1,\ y=2$

2) $x=1,\ y=1,\ z=2$

3) $x=1,\ y=1,\ z=1,\ t=1$

4) a. The system has a unique solution only if $k\neq 1$, $k\neq -4$

b. $k=-2$

c. No value of k

d. Proved as shown in the video

The Adjoint Matrix

Questions

1) Given the 2×2 matrix $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$. Compute $adj(A)$ and use it to compute A^{-1}.

2) Given the 3×3 matrix $A=\begin{bmatrix}2 & 1 & 1\\ 0 & 2 & -1\\ 5 & 2 & 3\end{bmatrix}$. Compute $adj(A)$ and use it to compute A^{-1}.

3) Given the 4×4 matrix $A=\begin{bmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 1 & 1\\ 0 & 0 & 1 & 1\end{bmatrix}$. Compute $adj(A)$ and use it to compute A^{-1}

4) Given the 5×5 matrix $A=\begin{bmatrix}-9 & 26 & -1 & 14 & 10\\ 13 & -7 & 87 & 4 & 0\\ 71 & 35 & 3 & 0 & 0\\ 17 & 2 & 0 & 0 & 0\\ 2 & 0 & 0 & 0 & 0\end{bmatrix}$.

Compute $ad(A)_{1,5}$, and use it to compute $\left(A^{-1}\right)_{1,5}$.

5) Solve the following:

a. Let A be a square matrix with $det(A)=1$. Prove that if all the elements of A are integers, then the elements of A^{-1} are integers too.

b. Let A be an invertible lower-triangular matrix. Prove that A^{-1} is also lower-triangular.

c. Prove that if A is invertible then so are $adj(A)$ and A^T.

d. If matrices A,B are invertible but C,D aren't, which of these is invertible:

i. $C+D$ ii. $A+B$ iii. AD iv. CD v. AB

6) Find the values of k for which the following matrix is non-invertible:

$$\begin{bmatrix} 4 & 0 & 7 & 5 & 0 \\ 0 & 0 & 3k & 0 & 0 \\ -7k^2 & 2 & 4k & k & 9+k \\ 3 & 0 & 4 & 2 & -1 \\ -5 & 0 & -8 & -3 & 2 \end{bmatrix}$$

Answer Key

1) $\begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix}$

2) 1

3) -1

4) $\frac{1}{2}$

5) a-c. Solution in the recording. d. AB

6) The matrix is non-invertible if and only if $k = 0$.

Geometrical Applications of Determinants

Questions

1) Answer the following:

a. Compute the area of the parallelograms whose vertices are given below:

i. $(0,0),(5,2),(6,5),(11,6)$ ii. $(-1,0),(0,5),(1,-4),(2,1)$

b. Compute the volume of the tetrahedron with vertices:

$(0,0,0),(1,0,-2),(1,2,4),(7,1,0)$

c. Find the equation of the plane passing through the points:

$(3,3,-2)$, $(-1,3,1)$, $(1,1,-1)$

d. Compute the area of the triangle with vertices: $(1,2),(3,4),(5,8)$.

Answer Key

1) a. i. 13; ii. 14
b. 22
c. $3x-y+4z+2=0$
d. 2