proprep

# Workbook

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# Matrices

## Matrix and Basic Operations on Matrices

### Questions

**1)** For each of the following matrix "calculations", say whether it's undefined or give the order [size] of the result:

a. $A_{4\times6} + B_{4\times6}$

b. $A_{4\times6} \cdot B_{4\times6}$

c. $A_{4\times6} \cdot C_{6\times2} - D_{4\times2}$

d. $A_{4\times6} \cdot E_{6\times4} - B_{4\times6}$

e. $B_{4\times6} + A_{4\times6} \cdot B_{4\times6}$

f. $E_{6\times4}\left(B_{4\times6} + A_{4\times6}\right)$

g. $\left(E_{6\times4} + A^T_{4\times6}\right)D_{4\times2}$

h. $E^T_{6\times4} \cdot B_{4\times6}$

i. $E_{6\times4} \cdot A_{4\times6} \cdot C_{6\times2}$

j. $E_{6\times4}\left(B_{4\times6} - A_{4\times6}\right)$

**2)** Solve the matrix equation $\begin{bmatrix} x+2y & 3x-2y \\ 2x-5y & 2x+8y \end{bmatrix} = \begin{bmatrix} 2-2z & 5+z \\ -4-3z & 3-12z \end{bmatrix}$, for $x$, $y$, $z$.

**3)** Given the matrices $C = \begin{bmatrix} 1 & 4 & 2 \\ 4 & 1 & 5 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 4 & 2 \\ 1 & 0 & -1 \\ 4 & 2 & 10 \end{bmatrix}$, $E = \begin{bmatrix} 4 & 1 & 1 \\ -1 & 0 & 1 \\ 4 & 1 & -1 \end{bmatrix}$. Evaluate:

a. $E + D$

b. $E - D + I_3$

c. $5C$

d. $2D + 4E \cdot I_3$

e. $2tr\left(D^2 - 2E\right)$

f.

Given the matrices $A = \begin{bmatrix} 4 & 0 \\ 1 & 2 \\ -1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 1 \\ 0 & -2 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 4 & 2 \\ 4 & 1 & 5 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 4 & 2 \\ 1 & 0 & -1 \\ 4 & 2 & 10 \end{bmatrix}$.

g. $4C^T + A$

h. $\frac{1}{2}A^T + \frac{1}{4}C$

i. $I_2 \cdot B \cdot C$

j. $trC^TC$

k. $D \cdot A \cdot B \cdot C$

l.

**4)** Rewrite each of the following SLEs in matrix form $A\underline{x}=\underline{b}$:

a. $\begin{cases} 2x+y-z=3 \\ x+2y-4z=5 \\ 6x+4y+z=2 \end{cases}$

b. $\begin{cases} 2x-3y+z+t=1 \\ 4x+y+2z=4 \\ y+z+t=1 \\ x-4z-2y=10 \end{cases}$

**5)** Given: $A=\begin{bmatrix} 4 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & -6 & 3 \end{bmatrix}$ $\underline{x}=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $\underline{b}=\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Express each of the following as an SLE:

a. $A\underline{x}=\underline{b}$

b. $A\underline{x}=4\underline{x}+\underline{b}$

c. $A\underline{x}=-k\underline{x}+\underline{b}$

d. $A\underline{x}=\underline{x}$

e. $A^T\underline{x}=2\underline{x}+3\underline{b}$

f.

**6)** Recall that a square matrix $A$ is *symmetric* if $A^T=A$ and *antisymmetric* if $A^T=-A$.

a. Given a square matrix $A$, which of the following must be true?

i. $A\cdot A^T$ is symmetric

ii. $A+A^T$ is symmetric

iii. $A-A^T$ is antisymmetric

b. Given 2 antisymmetric $n\times n$ matrices $A$ and $B$, which of the following must be true:

i. $BABABA$ is antisymmetric

ii. $A^2-B^2$ is symmetric

iii. $A^2+B$ is symmetric

c. Given 2 antisymmetric $n\times n$ matrices $A$ and $B$, which satisfy $AB=-BA$.
Which of the following must be true:

i. $AB^3$ is antisymmetric

ii. $AB^2$ is symmetric

iii. $(A-B)^2$ is symmetric

d. Given 2 antisymmetric $n\times n$ matrices $A$ and $B$, which satisfy $AB=BA$. Prove:

i. $AB$ is antisymmetric

ii. $AB+B$ is antisymmetric

e. Given that the $n\times n$ matrices $A$, $B$, $AB$ are symmetric, prove that $A^4B^4=B^4A^4$.

## Answer Key

**1)** a. $C_{4\times6}$ b. Undefined. c. $B_{4\times2}$ d-e. Undefined.

f. $C_{6\times6}$ g. $B_{6\times6}$ h. Undefined. i. $B_{6\times2}$

j. $C_{6\times6}$

**2)** $x=2,\ y=1,\ z=-1$

**3)** a. $\begin{bmatrix}5&5&3\\0&0&0\\8&3&9\end{bmatrix}_{3\times3}$ b. $\begin{bmatrix}4&-3&-1\\-2&1&2\\0&-1&-10\end{bmatrix}_{3\times3}$ c. $\begin{bmatrix}5&20&10\\20&5&25\end{bmatrix}$ d. $\begin{bmatrix}18&12&8\\-2&0&2\\24&8&16\end{bmatrix}$

e. 230 f. $\begin{bmatrix}8&16\\17&6\\7&21\end{bmatrix}_{3\times2}$ g. $\begin{bmatrix}2\frac{1}{4}&1\frac{1}{2}&0\\1&1\frac{1}{4}&1\frac{3}{4}\end{bmatrix}$ h. $\begin{bmatrix}8&17&13\\-8&-2&-10\end{bmatrix}_{2\times3}$

i. 63 j. $\begin{bmatrix}-32&82&-22\\48&87&75\\-48&108&-36\end{bmatrix}$

**4)** a. $\underbrace{\begin{bmatrix}2&1&-1\\1&2&-4\\6&4&1\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}x\\y\\z\end{bmatrix}}_{\underset{\sim}{x}}=\underbrace{\begin{bmatrix}3\\5\\2\end{bmatrix}}_{\underset{\sim}{b}}$ b. $\underbrace{\begin{bmatrix}2&-3&1&1\\4&1&2&0\\0&1&1&1\\1&-2&-4&0\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}x\\y\\z\\t\end{bmatrix}}_{\underset{\sim}{x}}=\underbrace{\begin{bmatrix}1\\4\\1\\10\end{bmatrix}}_{\underset{\sim}{b}}$

**5)** a. $\begin{cases}4x-2y+4z=1\\x-y+z=2\\x-6y+3z=3\end{cases}$ b. $\begin{cases}-2y+4z=1\\x-5y+z=2\\x-6y-z=3\end{cases}$ c. $\begin{cases}(4+k)x-2y+4z=1\\x+(-1+k)y+z=2\\x-6y+(3+k)z=3\end{cases}$

d. $\begin{cases}3x-2y+4z=0\\x-2y+z=0\\x-6y+2z=0\end{cases}$ e. $\begin{cases}2x+y+z=3\\-2x-3y-6z=6\\4x+y+z=9\end{cases}$

**6)** a. All three are true. b. (ii) Is true. c. All three are true.

d-e. Solution in the recordings.

## Matrix Inverse and its Applications

## Questions

Find the inverse of the matrix:

**1)** $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$

**2)** $A=\begin{bmatrix}5 & 2\\ 7 & 3\end{bmatrix}$

**3)** $A=\begin{bmatrix}4 & 1.5\\ 2 & 1\end{bmatrix}$

**4)** $A=\begin{bmatrix}1 & 0 & 2\\ 4 & -1 & 8\\ 2 & 1 & 3\end{bmatrix}$

**5)** $A=\begin{bmatrix}2 & 1 & 1\\ 0 & 2 & -1\\ 5 & 2 & 3\end{bmatrix}$

**6)** $A=\begin{bmatrix}2 & -1 & 1\\ 3 & -2 & 2\\ 5 & -3 & 4\end{bmatrix}$

**7)** $A=\begin{bmatrix}1 & 0 & 0 & 0\\ 1 & 2 & 0 & 0\\ 1 & 2 & 3 & 0\\ 1 & 2 & 3 & 4\end{bmatrix}$

Find the values of $k$ for which the following matrixes are invertible:

**8)** $\begin{bmatrix}1 & -1 & 1\\ 5 & -7 & k^2+3\\ 3 & -1 & k+3\end{bmatrix}$

**9)** $\begin{bmatrix}1 & 1 & 1 & 1 & k\\ 1 & 1 & 1 & k & 1\\ 1 & 1 & k & 1 & 1\\ 1 & k & 1 & 1 & 1\\ k & 1 & 1 & 1 & 1\end{bmatrix}$

**10)** Solve the following SLE by using the Inverse Matrix method: $\begin{cases}2x-y+z=3\\ 3x-2y+2z=5\\ 5x-3y+4z=11\end{cases}$.

**11)** Solve the following SLE by using the Inverse Matrix method: $\begin{cases}x+4y+2z+4t=1\\ x+2y-z=0\\ y+z+t=1\\ x+3y-z-2t=0\end{cases}$.

## Answer Key

**1)** $\begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix}$

**2)** $\begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$

**3)** $\begin{bmatrix} 1 & -1.5 \\ -2 & 4 \end{bmatrix}$

**4)** $\begin{bmatrix} -11 & 2 & 2 \\ 4 & -1 & 0 \\ 6 & -1 & -1 \end{bmatrix}$

**5)** $\begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ -10 & 1 & 4 \end{bmatrix}$

**6)** $\begin{bmatrix} 2 & -1 & 0 \\ 2 & -3 & 1 \\ -1 & -1 & 1 \end{bmatrix}$

**7)** $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{4} & \frac{1}{4} \end{bmatrix}$

**8)** $k \neq -2$ , $k \neq 1$

**9)** $k = 1$ , $k = -4$

**10)** $x = 1,\ y = 2,\ z = 3$

**11)** $x = -13,\ y42,\ z = -5,\ t = 2$

## Properties of the Matrix Inverse

## Questions

**1)** Assume that all the matrices are invertible and of order $n$, and extract $X$:

a. $AXC = D$ b. $A^{-1}XC = A^{-1}DC$ c. $P^{-1}X^{T}P = A$

d. $C^{-1}(A+X)D^{-2} = I$ e. $(A-AX)^{-1} = X^{-1}C$ f. $ABC^{T}X^{-1}BA^{T}C = AB^{T}$

**2)** Given that $B = \begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix}$ and that $B^{2}X(2B)^{-1} = B+I$. Find $X$. All matrices are $2\times2$.

**3)** Given that $B^{-1} = \begin{bmatrix} 1 & 0 & 2 \\ 4 & -1 & 8 \\ 2 & 1 & 3 \end{bmatrix}$ and that $BYB^{T} = B^{-1} + B$. Find $Y$. All matrices are $3\times3$.

**4)** Given that $A^{-1} = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}$ and that $5A^{T}\mathrm{B}(\mathrm{I}+2\mathrm{A})^{-2} = (7A)^{-2}$. Find $B$. All matrices are $2\times2$.

**5)** Prove the following:

a. Given: $A$ is a square matrix satisfying $A^{2} - 5A - 2I = 0$.
Prove that $A$ is invertible and express $A^{-1}$ in terms of $A$ and $I$.

b. Given: $A$ is a square matrix satisfying $(A-3I)(A+2I) = 0$.
Prove that $A$ is invertible and express $A^{-1}$ in terms of $A$ and $I$.

**6)** Prove the following:

a. Given: $A$ is a square matrix satisfying $A^{2} - 5A - 2I = 0$.
Prove that $A$ is invertible and express $A^{-1}$ in terms of $A$ and $I$.

b. Given: $A$ is a square matrix satisfying $(A-3I)(A+2I) = 0$.
Prove that $A$ is invertible and express $A^{-1}$ in terms of $A$ and $I$.

**7)** Given: $A=\begin{bmatrix}-1 & 3 & 0\\ 3 & -1 & 0\\ -2 & -2 & 6\end{bmatrix}$, $p(x)=x^3-4x^2-20x+48$.

a. Compute $p(A)$.

b. Using part a. above, prove that $A$ is invertible and express $A^{-1}$ in terms of $A$ and $I$.

**8)** Given: $A$ is a square matrix satisfying $A^4=0$.

a. Prove that $A$ is noninvertible.

b. Prove that the matrix $I-A$ is invertible and find its inverse.

**9)** Given: $\begin{cases}P^{-1}AP=B\\ Q^{-1}BQ=C\end{cases}$. Prove that there exists an invertible matrix $D$ such that $D^{-1}AD=C$.

## Answer Key

**1)** a. $X = A^{-1}DC^{-1}$ b. $X = D$ c. $X = \left(P^{-1}\right)^{T} A^{T} P^{T}$

d. $X = CD^{2} - A$ e. $X = \left(A + C^{-1}\right)^{-1} A$ f. $X = BA^{T}C \cdot \left(B^{T}\right)^{-1} BC^{T}$

**2)** $X = \begin{bmatrix} 20 & -4 \\ -8 & 4 \end{bmatrix}$

**3)** $Y = \begin{bmatrix} 22 & 86 & 38 \\ 64 & 246 & 114 \\ 56 & 222 & 92 \end{bmatrix}$

**4)** $B = \frac{1}{245}\begin{bmatrix} 264 & 450 \\ 448 & 768 \end{bmatrix}$

**5)** Solution in the video.

**6)** Solution in the video.

**7)** a. $P(A) = 0$ b. Solution in the video.

**8)** Solution in the video.

**9)** Solution in the video.

## Elementary Matrices and LU Decomposition

## Questions

**1)** Write the LU decomposition of the matrix $A=\begin{bmatrix} 1 & 0 & 2 \\ 4 & -1 & 8 \\ 2 & 1 & 6 \end{bmatrix}$.

**2)** Write the LU decomposition of the matrix $A=\begin{bmatrix} 1 & 2 & -3 & 4 \\ 2 & 3 & -8 & 5 \\ 1 & 3 & 1 & 3 \\ 3 & 8 & -1 & 13 \end{bmatrix}$.

*For the solution see the video.