

# Workbook



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# N<sup>th</sup> Order Linear Equations

## Linear, Homogeneous, Constant Coefficients

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### Questions

Solve the following equations:

1)  $y''' - 2y'' - 3y' = 0$

3)  $y''' - 2y'' - y' + 2y = 0$

5)  $y^{(4)} - y = 0$

7)  $y^{IV} + y = 0$  [ $y^{IV} = y^{(4)} = y''''$ ]

9)  $(D^5 + 3D^4 + 2D^3 - 2D^2 - 3D - 1)y = 0$

11)  $z''' - 6z'' + 12z' - 8z = 0$

13)  $x^{(6)} - 3x^{(4)} + 3x'' - x = 0$ ,  $x = x(t)$

15) 
$$\begin{cases} y'''' - 3y''' + 6y'' - 12y' + 8y = 0 \\ y(0) = 2, y'(0) = 5, y''(0) = -19, y'''(0) = -47 \end{cases}$$

2)  $y^{(4)} + 3y''' - 15y'' - 19y' + 30y = 0$

4)  $y^{(4)} - 5y'' + 4y = 0$

6)  $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 20y = 0$

8)  $y^{(6)} - y'' = 0$

10)  $y^{(8)} + 8y^{(4)} + 16y = 0$

12)  $y^{(4)} - 4y = 0$

14) 
$$\begin{cases} y''' - y'' + y' - y = 0 \\ y(0) = 3, y'(0) = 4, y''(0) = -1 \end{cases}$$

- 16) Given a homogeneous ODE of the 6<sup>th</sup> order with constant coefficients, one of whose solutions is  $x^2e^x \cos 2x$ .
- Find the general solution of the equation.
  - Find the equation.

### Answer Key

- 1)  $y = c_1 + c_2e^{-x} + c_3e^{3x}$
- 2)  $y = c_1e^x + c_2e^{-2x} + c_3e^{3x} + c_4e^{-5x}$
- 3)  $y = c_1e^{2x} + c_2e^x + c_3e^{-x}$
- 4)  $y = c_1e^x + c_2e^{-x} + c_3e^{2x} + c_4e^{-2x}$
- 5)  $y = c_1e^x + c_2e^{-x} + c_3 \cos x + c_4 \sin x$
- 6)  $y = c_1e^{-4x} + e^x[c_2 \cos 2x + c_3 \sin 2x]$
- 7)  $y = e^{\frac{\sqrt{2}}{2}x} \left( c_1 \cos \frac{\sqrt{2}}{2}x + c_2 \sin \frac{\sqrt{2}}{2}x \right) + e^{-\frac{\sqrt{2}}{2}x} \left( c_3 \cos \frac{\sqrt{2}}{2}x + c_4 \sin \frac{\sqrt{2}}{2}x \right)$
- 8)  $y = c_1 + c_2x + c_3e^x + c_4e^{-x} + \cos x + \sin x$
- 9)  $y = c_1e^x + c_2e^{-x} + c_3xe^{-x} + c_4x^2e^{-x} + c_5x^3e^{-x}$
- 10)  $+ e^{-x}[c_5 \cos x + c_6 \sin x] + xe^{-x}[c_7 \cos x + c_8 \sin x]$
- 11)  $y = c_1e^{2x} + c_2xe^{2x} + c_3x^2e^{2x}$
- 12)  $y = c_1e^{\sqrt{2}x} + c_2e^{-\sqrt{2}x} + c_3 \cos \sqrt{2}x + c_4 \sin \sqrt{2}x$
- 13)  $y = c_1e^t + c_2te^t + c_3t^2e^t + c_4e^{-t} + c_5te^{-t} + c_6t^2e^{-t}$
- 14)  $y = e^x + 2 \cos x + 3 \sin x$
- 15)  $y = e^x - 2e^{2x} + 3 \cos 2x + 4 \sin 2x$
- 16) a.  $y = e^x[c_1 \cos 2x + c_2 \sin 2x] + xe^x[c_3 \cos 2x + c_4 \sin 2x] + x^2e^x[c_5 \cos 2x + c_6 \sin 2x]$   
b.  $y^{(6)} - 6y^{(5)} + 27y^{(4)} - 68y''' + 135y'' - 150y' + 125y = 0$

### Method of Undetermined Coefficients

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#### Questions

Solve the following equations:

1)  $y''' - 2y'' - 3y' = 2\sin x - 4\cos x$

2)  $y^{(4)} + 3y''' - 15y'' - 19y' + 30y = -28e^{2x}$

3)  $y''' - 2y'' - y' + 2y = 2x^3 - 3x^2 - 12x + 14$

4)  $y''' - 3y' + 2y = e^x$

5)  $y''' - y'' + y' - y = \sin x$

#### Answer Key

1)  $y = c_1 + c_2e^{-x} + c_3e^{3x} + \sin x$

2)  $y = c_1e^x + c_2e^{-2x} + c_3e^{3x} + c_4e^{-5x} + e^{2x}$

3)  $y = c_1e^{2x} + c_2e^x + c_3e^{-x} + x^3 + 4$

4)  $y = c_1e^x + c_2xe^x + c_3e^{-2x} + \frac{1}{6}x^2e^x$

5)  $y = c_1e^x + c_2\cos x + c_3\sin x + \frac{1}{4}x(\cos x - \sin x)$

### The Wronskian and its Uses

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#### Questions

Solve the following equations:

- 1) Is it possible that  $y_1(x) = x$ ,  $y_2(x) = x^2$ ,  $y_3(x) = x^3$  are three solutions of a linear homogeneous 3<sup>rd</sup> order ODE  $y''' + p(x)y'' + q(x)y' + r(x)y = 0$ , with continuous coefficients  $p(x)$ ,  $q(x)$ ,  $r(x)$  on the interval  $[0, \pi]$ ?
- 2) We're given the functions  $y_1(x) = 4 - x$ ,  $y_2(x) = 4 + x$ ,  $y_3(x) = 20 + x$ .
  - a. Compute the Wronskian of the functions.
  - b. Determine whether or not the functions are l.i. on  $(-\infty, \infty)$ .
  - c. Answer b. again, after noting that the three functions are solutions of  $y'' = 0$ .
- 3) Answer the following Questions
  - a. Let  $y_1(x)$ ,  $y_2(x)$ ,  $y_3(x)$  be functions which are thrice continuously differentiable on an interval  $I$ , and such that their Wronskian is nonzero on  $I$ . Prove that there exists an ODE  $y''' + p(x)y'' + q(x)y' + r(x)y = 0$  with continuous coefficients on  $I$ , such that  $y_1(x)$ ,  $y_2(x)$ ,  $y_3(x)$  are three of its solutions.
  - b. Find an equation  $y''' + p(x)y'' + q(x)y' + r(x)y = 0$  with continuous coefficients on  $x > 0$  such that  $y_1(x) = x$ ,  $y_2(x) = x^2$ ,  $y_3(x) = x^3$  are three of its solutions.

#### Answer Key

- 1)  $y_1(x) = x$ ,  $y_2(x) = x^2$ ,  $y_3(x) = x^3$ , can't possibly be solutions of the ODE as described.
- 2) a.  $W = 0$       b. Thus, the functions are linearly dependent.  
c. When the functions are solutions of this kind of ODE the problem becomes easier.  
We can immediately conclude from  $W = 0$  that they are linearly dependent.
- 3) a.  $y''' - y'' \frac{|D|}{W} + y' \frac{|Q|}{W} - y \frac{|A|}{W} = 0$   
b.  $y''' - \frac{3}{x}y'' + \frac{6}{x^2}y' - \frac{6}{x^3}y = 0$ , ( $x > 0$ )