

Workbook



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Phase Planes

Phase Planes

Questions

- 1) For the matrix: $A = \begin{bmatrix} -4 & 1 \\ 3 & -2 \end{bmatrix}$, consider the ODE system: $x' = Ax$, where $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$

denotes the position of a particle in the plane at time t .

- Find the general solution $x(t)$ of this differential equation.
- Classify the critical point and sketch a phase portrait of the system in the plane.

- 2) For the matrix: $A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$, consider the ODE system: $x' = Ax$, where $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$

denotes the position of a particle in the plane at time t .

- Find the general solution $x(t)$ of this differential equation.
- Classify the critical point and sketch a phase portrait of the system in the plane.

- 3) For the matrix: $A = \begin{bmatrix} 4 & 3 \\ 8 & 2 \end{bmatrix}$, consider the ODE system: $x' = Ax$, where $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$

denotes the position of a particle in the plane at time t .

- Find the general solution $x(t)$ of this differential equation.
- Classify the critical point and sketch a phase portrait of the system in the plane.

- 4) For the nonsingular matrix: $A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$,

consider the ODE system: $x' = Ax$, where $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$.

- Find the general solution $x(t)$ of the system.
- Classify the critical point (at the origin) and sketch a portrait of the system in the phase plane.

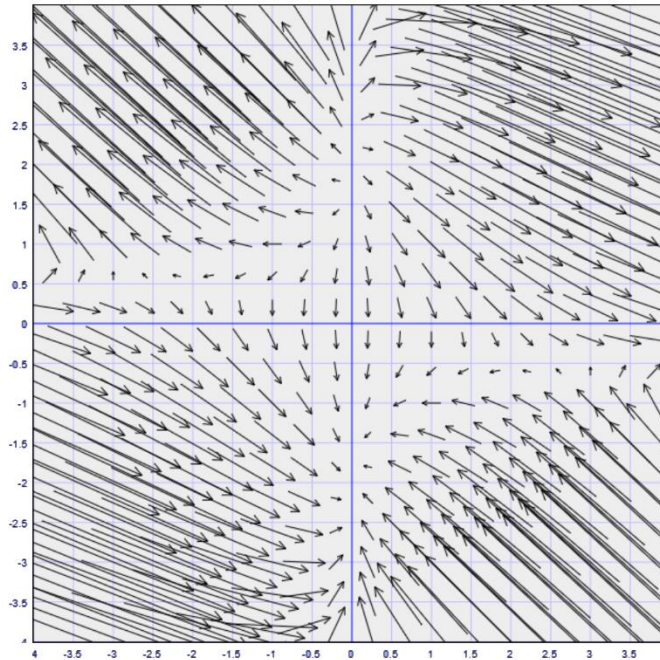
- 5) Consider the nonlinear planar system of ODEs:
$$\begin{cases} \frac{dx}{dt} = x(4 - x - 3y) \\ \frac{dy}{dt} = y(3 - x - 2y) \end{cases}.$$
- Find the critical points (AKA equilibrium solutions).
 - Near each critical point, find the corresponding linear system and determine the type of its critical point (of the linear system).
 - Which critical points of the original system are stable? Unstable? Asymptotically stable?
 - Try to give a rough sketch of the phase portrait for the system.
 - For $x(t) \geq 0, y(t) \geq 0$, suppose the system models the populations $x(t), y(t)$ of two species.
Does the model suggest the species could coexist in a stable way?
- 6) Let A be a 2×2 real matrix with eigenvalues $\lambda_1 = -1, \lambda_2 = -5$ and corresponding eigenvectors $v_{\lambda_1=-1} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, v_{\lambda_2=-5} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.
- Give a general solution to the linear planar system: $x' = Ax$, where $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$.
 - Find an invertible matrix T and a diagonal matrix D , such that $T^{-1}AT = D$.
 - Compute e^{tA} .
 - Sketch a phase portrait (with arrows) and classify the type and stability of the origin. [Recall that the origin is the only critical point of such a system]
- 7) Consider the nonlinear planar system of autonomous ODEs
$$\begin{cases} x' = 2xy = F(x, y) \\ y' = 2x - y^2 + 9 = G(x, y) \end{cases}.$$
- Find a nonlinear function $H(x, y)$, such that every orbit (trajectory) of the system satisfy $H(x, y) = c$ for some constant c .
 - Find the critical points.
Which of these points are saddle points?
 - For each value of c that corresponds to a saddle point, sketch the level set $H(x, y) = c$, plus the critical points.

Ordinary Differential Equations

- 8) Consider the nonlinear planar system of autonomous ODEs $\begin{cases} x' = -3x + y = F(x, y) \\ y' = -5x - y + 4x^2 = G(x, y) \end{cases}$.
- Find the stationary (critical) points of the system.
 - Find the Jacobian matrix at each stationary point.
 - Classify the type and stability of each stationary point.
 - Sketch a phase portrait of the system, that shows its behavior near each stationary point (Use arrows to mark the sketched orbits).

- 9) Consider the nonlinear planar system of ODEs: $\begin{cases} \frac{dx}{dt} = \underbrace{5xy + x^2}_{F(x,y)} \\ \frac{dy}{dt} = \underbrace{y^2 - 3xy - 4}_{G(x,y)} \end{cases}$

and the direction field plot (by MATLAB), as shown:



- What are the stationary (critical) points of the system?
- Classify the type and stability of each stationary point.
- Predict the behavior, as $t \rightarrow \infty$, of the solution passing through $(2.5, 0)$. Sketch your predicted solution in the phase plane.

10) Consider the nonlinear autonomous system $\begin{cases} x' = x^2 - 9 = F(x, y) \\ y' = 2xy = G(x, y) \end{cases}$.

- Find the critical points of the system.
- Write down the linearization of the system at each critical point.
- Classify the type and stability of each critical point.
- Sketch a phase portrait of the system (with arrows on the sketched orbits).

11) Consider the ODE system: $x' = \begin{bmatrix} 9 & 2 \\ -8 & 1 \end{bmatrix}x$, where $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$.

Classify the critical point and sketch a phase portrait of the system in the plane.



Answer Key

- 1) a. $x(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 e^{-5t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ b. $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ node point which is asymptotically stable.
- 2) a. $x(t) = c_1 e^{-2t} \begin{bmatrix} -2 \\ 3 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ b. $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ saddle point which is unstable.
- 3) a. $x(t) = c_1 e^{-2t} \begin{bmatrix} -1 \\ 2 \end{bmatrix} + c_2 e^{8t} \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ b. $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ saddle point which is unstable.
- 4) a. $x(t) = c_1 e^t \begin{bmatrix} \cos 2t \\ -\sin 2t \end{bmatrix} + c_2 e^t \begin{bmatrix} \sin 2t \\ \cos 2t \end{bmatrix}$ b. $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ spiral point which is unstable.
- 5) a. $(0,0)$, $\left(0, \frac{3}{2}\right)$, $(4,0)$, $(1,1)$
- b. We find: $A(x,y) = \begin{bmatrix} 4-2x-3y & -3x \\ -y & 3-x-4y \end{bmatrix}$. Then:
- $A(0,0) = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$ unstable node. $A\left(0, \frac{3}{2}\right) = \begin{bmatrix} -\frac{1}{2} & 0 \\ -\frac{3}{2} & -3 \end{bmatrix}$, asymptotically stable node.
- $A(4,0) = \begin{bmatrix} -4 & -12 \\ 0 & -1 \end{bmatrix}$, asymptotically stable node. $A(1,1) = \begin{bmatrix} -1 & -3 \\ -1 & -2 \end{bmatrix}$, saddle (unstable).
- c. $(0,0)$, $(1,1)$ are unstable, $\left(0, \frac{3}{2}\right)$, $(4,0)$ are asymptotically stable.
- d. Solution in the recording. e. No.
- 6) a. $x(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 e^{-5t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ b. $T = \begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix}$, $D = \begin{bmatrix} -1 & 0 \\ 0 & -5 \end{bmatrix}$
- c. $e^{tA} = \frac{1}{4} \begin{bmatrix} e^{-t} + 3e^{-5t} & e^{-t} - e^{-5t} \\ 3e^{-t} - 3e^{-5t} & 3e^{-t} + e^{-5t} \end{bmatrix}$ d. Solution in the recording.
- 7) a. $H(x,y) = x^2 - y^2 x + 9x$ b. $(4.5,0)$ Circle point. $(0, \pm 3)$ Saddle points.
- c. Solution in the recording.

Ordinary Differential Equations

- 8)** a. $(0,0)$, $(2,6)$ b. $A(0,0) = \begin{bmatrix} -3 & 1 \\ -5 & -1 \end{bmatrix}$, $A(2,6) = \begin{bmatrix} -3 & 1 \\ 11 & -1 \end{bmatrix}$
- c. $(0,0)$ spiral point, asymptotically stable. $(2,6)$ saddle point, unstable.
- d. Solution in the recording.
- 9)** a. $(0,2)$, $(0,-2)$, $(-2.5,0.5)$, $(2.5,-0.5)$
- b. $(0,2)$ node, unstable. $(0,-2)$ node, asymptotically stable.
- $(-2.5,0.5)$ saddle, unstable. $(2.5,-0.5)$ saddle, unstable.
- c. Solution in the recording.
- 10)** a. $(3,0)$, $(-3,0)$
- b. Linearization at $(3,0)$: $x' = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} x$ Linearization at $(-3,0)$: $x' = \begin{bmatrix} -6 & 0 \\ 0 & -6 \end{bmatrix} x$.
- c. $(3,0)$ proper node, unstable, $(-3,0)$ proper node, asymptotically stable.
- d. Solution in the recording.
- 11)** Solution in the recording.