proprep™

Workbook

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Ordinary Differential Equations (ODE)

Separation of Variables

Questions

Solve the following equations:

1) $\frac{dy}{dx}=\frac{x^2}{y}$, $(y\neq 0)$

2) $(1-x)y'=y^2$

3) $yy'\sqrt{1+x^2}+x\sqrt{1+y^2}=0$

4) $(x-1)\frac{dy}{dx}=4y$; $y(2)=1$

5) $\frac{dy}{dx}=xy+3y-3x-9$; $y(1)=-1$

6) $(x^2y-2+2x^2-y)dx-(xy^2-4-4x+y^2)dy=0$

7) $dy=2t(y^2+4)dt$

8) $\frac{dx}{dt}=x^2-2x+2$

9) $y'+y^2\sin x=0$; $y(\pi)=1$

10) $\frac{dy}{dx}=y\sec^2 x$; $y(0)=5$

11) $\frac{dy}{dx}=\frac{xy^3}{\sqrt{1+x^2}}$; $y(0)=1$

12) Let $y(t)$ denote the amount of substance (or population) that is either growing or decaying. Assume that the time rate of change of this amount of substance is proportional to the amount of substance present (i.e. the amount $y(t)$ grows/decays exponentially). Assume that at the start time $t=0$ the amount is y_0, and find a formula for the amount at any given time t.

13) If the population of the earth was found to be 4 billion in 1980, and is increasing at a rate of 2% per year.

a. Find the population of the earth in 2010.
b. Find the population of the earth in 1974.

When will a population of 50 billion be reached?

Answer Key:

1) $y = \pm\sqrt{\frac{2}{3}x^3 + k}$

2) $y = \frac{1}{\ln|1-x| - c}$, $y = 0$

1) $\sqrt{1+y^2} = -\sqrt{1+x^2} + c$

2) $\frac{1}{4}\ln|y| = \ln|x-1|$, $(x \neq 1)$

3) $\ln|y-3| = \frac{x^2}{2} + 3x + \ln 4 - 3.5$, $(y \neq 3)$

4) $\frac{x^2}{2} - x = \frac{y^2}{2} - 2y + c$, $y = -2$

5) $y = 2\tan\left(2t^2 + k\right)$

6) $x = 1 + \tan(t + c)$

7) $y = -\frac{1}{\cos x}$

8) $\ln|y| = \tan x + \ln 5$

9) $\frac{1}{-2y^2} = \sqrt{1+x^2} - 1.5$

10) $y(t) = y_o e^{kt}$

11) a. 7.28 billion b. 4.51 billion c. at 2016.

Homogeneous Equations

Questions

Solve the following equations:

1) $\left(y^3+x^3\right)dx+xy^2dy=0$

2) $y'=\dfrac{4y-3x}{2x-y}$

3) $y^2+x^2y'=xyy'$

4) $\left(3xy+y^2\right)dx+\left(x^2+xy\right)dy=0$

5) $\left(x-y\cos\dfrac{y}{x}\right)dx+x\cos\dfrac{y}{x}dy=0$

6) $y'=\dfrac{2xye^{(x/y)^2}}{y^2+y^2e^{(x/y)^2}+2x^2e^{(x/y)^2}}$

7) $\left(y+\sqrt{x^2+y^2}\right)dx-xdy=0\ ;\ y(1)=0$

8) $\left(2x^2t-2x^3\right)dt+\left(4x^3-6x^2t+2xt^2\right)dx=0$

9) Answer the following:

a. Find n so that the O.D.E. $\left(y^2+x^2\right)dx+xy^ndy=0$ will be homogeneous.

b. Solve the above O.D.E. for the value of n you found in section a.

Answer Key

1) $-\ln|x| = \frac{1}{6}\ln\left|2\left(\frac{y}{x}\right)^3 + 1\right| + C$

2) $\ln|x| = \frac{1}{4}\ln\left|\left(\frac{y}{x}\right) - 1\right| - \frac{5}{4}\ln\left|\frac{y}{x} + 3\right| + C \quad y = x\,,\, y = -x$

3) $-\ln|x| = \ln\left|\frac{y}{x}\right| - \left(\frac{y}{x}\right) + C, \quad y = 0$

4) $-\ln|x| = \frac{1}{4}\ln\left|2\left(\frac{y}{x}\right)^2 + 4\right| + C, \quad y = x\,,\, y = -2x$

5) $\ln|x| = -\sin\left(\frac{y}{x}\right) + C$

6) $\ln\left(1 + e^{\left(\frac{x}{y}\right)^2}\right) = \ln|y| + C, \quad y = 0$

7) $\ln x = \sinh^{-1}\left(\frac{y}{x}\right)\,, \; (x > 0)$

8) $\ln|t| = -\frac{1}{2}\ln\left|\left(\frac{x}{t}\right) - \left(\frac{x}{t}\right)^2\right| + C, \quad x(t) = t\,,\, x(t) = 0$

9) a. $n = 1$

b. $\ln|x| = -\frac{1}{4}\ln\left(1 + 2\left(\frac{y}{x}\right)^2\right) + C$

Linear Equations of the First Order

Questions

Solve the following equations:

1) $y'+2xy=4x$

2) $xy'=y+x^3+3x^2-2x\ ,\ (x\neq 0)$

3) $(x-2)y'=y+2(x-2)^3\ ,\ (x>2)$

4) $x^3y'+(2-3x^2)y=x^3\ ,\ (x>0)$

5) $\frac{dy}{dt}+y=2+2t\ ;\ y(0)=1$

6) $\frac{dy}{dx}+y\cot x=5e^{\cos x}\ ,\ (\sin x>0)$

7) $y'-2y\cot x=1\ ,\ (\sin x>0)$

8) $x^2z'+2xz=\cos x\ ;\ z(\pi)=0$

Answer Key

1) $y = 2 + Ce^{-x^2}$

2) $y = x\left[\frac{x^2}{2} + 3x - 2\ln x + C\right]$, $(x > 0)$

3) $y = (x-2)\left[x^2 - 4x + C\right]$

4) $y = \frac{1}{2}x^3 + C \cdot x^3 e^{\frac{1}{x^2}}$

5) $y = 2t + e^{-t}$

6) $y = \frac{1}{\sin x}\left[-5e^{\cos x} + C\right]$

7) $y = \sin^2 x\left[-\cot x + C\right]$

8) $z = \frac{\sin x}{x^2}$

Word Problems

Questions

1) For a given curve, the slope of the tangent at each point (x,y) on the curve is equal to $-\dfrac{x}{y}$. Find the equation of the curve.

2) Given a curve, in the first quadrant, which goes through the point $(1,3)$, and that the slope of its tangent at the point (x,y) equals $-\left(1+\dfrac{y}{x}\right)$. Find the equation of the curve.

3) Find the equation of the curve, whose normal at each point passes through the origin**.**

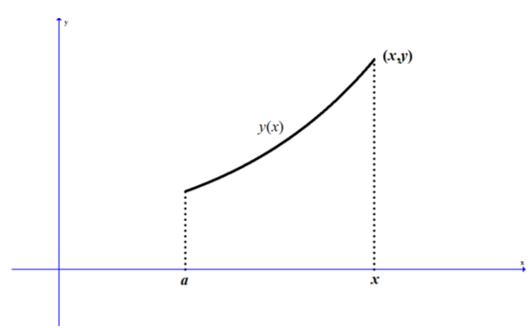
4) Find the equation of the curve, the slope of whose tangent at each point is equal to half the slope of the segment from the origin to the point.

5) Find the equation of the curve which passes through the point $(1,2)$ and for each point (x,y) on it, the slop of the normal is $\dfrac{2xy}{y^2-x^2}$.

6) Given a curve in the first quadrant, passing through the point $(2,4)$. Also given that for each point $\mathrm{A}(x,y)$ on it, the difference between the slope of the tangent to the curve at A and between the slope of the line connecting A with the origin, is equal to the y-coordinate of A. Find the equation of the curve.

7) Find the equation of the curve that passes through the origin and which is perpendicular to each line connecting a point on the curve to the point $(3,4)$.

8) The area S is bounded by the curve $y=y(x)$, the x-axis and the lines $x=a,\ x=x$ (variable); see diagram . It is known that the area S is proportional to the arc length between the points $(a,y(a))$ and $(x,y(x))$. Find the equation of the curve.



9) Find the family of curves orthogonal to the family $\{x+2y=c\}$.

10) Find the family of curves orthogonal to the family $\{xy=c\}$.

11) Answer the following questions:

a. Find the family of curves orthogonal to the family $\{x^2+2y^2=c\}$.

b. Find the curve orthogonal to the curve $x^2+2y^2=9$ at the point $(1,2)$ on it.

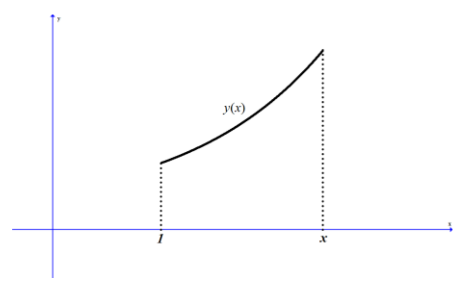
12) Find the family of curves orthogonal to the family $\{x^2+y^2=cx\}$.

13) Find the family of curves form a $45°$ angle with the family $\{x^2+y^2=c\}$.

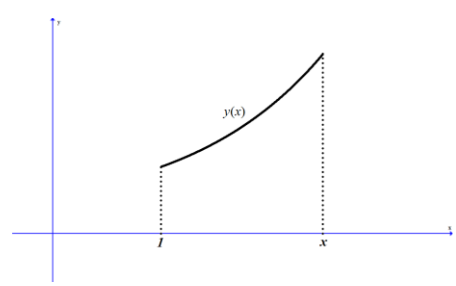
14) At each point on a curve the segment of the normal between the point and the x-axis is bisected by the y-axis. Find the equation of the curve.

15) Find the equation of the curve passing through the point $(0,1)$ such that the triangle bounded by the y-axis, the tangent to the curve at any point $\mathrm{M}(x,y)$ on it, and the segment OM from the origin O to M, is an isosceles triangle whose base is the segment MN, where N is the intersection of the tangent with the y-axis.
Illustrate the problem with a sketch in the first quadrant.

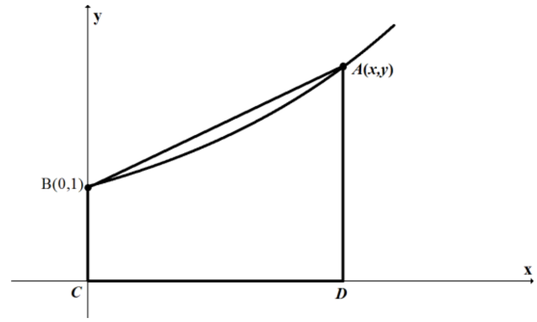
16) Area S is bounded by the curve $y=y(x)$ the x-axis and the lines $x=1,\ x=x$ (variable); See diagram.
It is known that $y(1)=2$.
Does such curve exist such that the area of S equals $2y(x)$?



17) Area S is bounded by the curve $y=y(x)$ the x-axis and the lines $x=1,\ x=x$ (variable); See diagram. It is known that $y(1)=2$.
Does such a curve exist such that the area of S equals $y(x)-2$?



18) Given a curve passing through the point $B(0,1)$. At each point A on the curve, the slope is equal to the area of trapezoid ABCD as shown in the figure.
What is the equation of the curve?



19) If a quantity $y(t)$ grows [decays] exponentially; i.e. at each instant the rate of growth [decay] is proportional to its value.
Suppose that at the start time $t=0$ the quantity is y_0 and that the constant of proportionality is k. Find a formula for the quantity at any time t.

20) The population of the earth is increasing at a rate of 2% per year.
It was found to be 4 billion in 1980.

a. What will the population of the earth be in 2010?
b. What was the population of the earth in 1974?
c. When will a population of 50 billion be reached?
(Assume that the population is growing exponentially; i.e. at each instant the rate of growth is proportional to its value).

21) The population in a certain city grows exponentially. In a certain year, there were 400 thousand residents and 4 years later there were 440 thousand.

a. Find the annual growth rate (as a $\%$).
b. After how many years (from that certain year) were there 550 thousand residents?

22) A man deposited money in the bank at an interest rate of 4% compounded annually.
After 5 years he had accumulated $\$5000$.

a. How much did he initially deposits?
b. After how many years will he have accumulated $\$7000$?

23) The number of wild animals at a nature reserve grows exponentially.
There were 1000 animals at the initial count.
At a second count, 20 months later, there were 1400 wild animals.
How many months after the initial count will the reserve have 2000 animals?

24) The radioactive isotope carbon-14 has a half-life of 5750 years.
At any given moment, its rate of decay is proportion to the amount present.

a. How many grams of this isotope will survive after 1000 years, if there were 100 grams initially?
b. After how many years will there remain just 10 grams of the initial 100 grams.

25) In a certain pool, there are 240 tons of fish, and the quantity of fish in it increases by 4% each week. In a second pool, there are 200 tons of fish, and the quantity of fish in it increases by 10% each week.

a. After how many weeks will both pools have the same quantity of fish?
b. After how many weeks will the second pool have twice the quantity of fish as the first pool?

26) At time $t=0$ a tank contains 4kg of salt dissolved in 200 liters of water. Salt water, at a concentration of 0.2kg per liter of water, is flowing into the tank at a rate of 25 liters per minute and, simultaneously, the mixed solution is draining out of the tank at the same rate.

a. Compute the amount of salt in the tank after 8 minutes.
b. After how long will the amount of salt in the tank be twice the initial amount?

27) A rowboat is initially towed at a rate of 12 km/h. At time $t=0$ the cable is released and a man in the boat starts rowing in the direction of the motion, and applies a force of 20 newton to the boat. The mass of the boat & rower is 500kg and the resistance (newton) is $2v$, where v is the velocity of the boat in meters/sec. Find:

a. The velocity of the boat after 30 second? In $\frac{m}{\sec}$
b. When the velocity of the boat will be $5\left[\frac{m}{\sec}\right]$?
c. The asymptotic velocity of the boat (i.e. as $t\to\infty$)

28) Newton's Law of Cooling states that the rate of change of the temperature of an object is proportional to the difference between its own temperature and temperature of its surroundings. A substance with a temperature of $150°C$ is in a container which has the surrounding temperature of the air, a constant $30°C$. The substance cools in accordance with Newton's Law of Cooling and, after half an hour, its temperature drops to $70°C$.

a. What is its temperature after an hour?
b. After how long will its temperature be $40°C$?

29) A spring of negligible weight is suspended vertically. A mass m is connected to its free end. If the mass is moving at a velocity V_0 m/sec when the spring is not extended, find the velocity V (in m/sec) as a function of the spring's extension (in m).

Answer Key

1) $y^2+x^2=k$

2) $2yx+x^2=7$

3) $x^2+y^2=k$

4) $y=k|x|$

5) $x^3-3y^2x=-11$

6) $y=2xe^{x-2}$

7) $y=4\pm\sqrt{25-(x-3)^2}$

8) $y=k\cosh\left(\pm\frac{1}{k}+C\right)$

9) $y=2x+k$

10) $y^2-x^2=k$

11) a. $y=ax^2$ b. $y=2x^2$

12) $y(t)=y_0e^{kt}$

13) $\ln|x|+\frac{1}{2}\ln\left(\left(\frac{y}{x}\right)^2+1\right)=-\arctan\left(\frac{y}{x}\right)+C$

14) $2x^2+y^2=k$

15) $2=y+\sqrt{y^2+x^2}$

16) No

17) Yes

18) $y=2e^{\frac{x^2}{4}}-1$

19) $y(t)=y_0e^{kx}$

20) a. 1.7.28bil b. 2.4.51bil c. year 2106

21) a. 2% b. 15.92 years.

22) a. 4093.65 b. 13.41 years.

23) 40.77 months.

24) 19188 years.

25) a. 3.04 years. b. 14.6

26) a. 26.75 kg b. 0.942 min

27) a. $4.09\frac{m}{\sec}$ b. $t=72\sec$ c. $10\frac{m}{\sec}$

28) a. $40°C$ b. $t=1.13$ hours

29) $V=\pm\sqrt{2gx-\frac{kx^2}{m}+V_0^{\,2}}$