proprep™

Workbook

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A Two-Dimensional Discrete Variable

Joint Probability Function

Questions

1) A person enters a casino with $\$75$ in his pocket. He plays on a slot machine where his chances of winning are 0.3. If he wins the game, the casino gives him $\$25$, and if he loses, he pays the casino $\$25$. The same person decides to stop playing the game if he has $\$100$, but in any case, he does not play more than three games.
We define X as the money he has when he leaves the casino and Y as the number of games he plays.

a. Construct the joint and marginal probability function.
b. What is the expectation of the number of games that he plays?
c. If he leaves the casino with $\$100$, what are the expectation and variance of the number of games he played?

2) The following is the joint and marginal probability function of two discrete variables:

a. Complete the missing probabilities in the table.
b. Are X and Y dependent?
c. Find the probability that $Y=3$, if it is known that $X=1$.

Y \ X	0	1	2	$\mathrm{P}(Y)$
2		0.08	0.12	0.4
3	0.1	0.05		
4				0.45
$\mathrm{P}(X)$		0.4	0.2	

3) A factory markets a product that is packed in packages of various sizes. The number of packages containing two products is the same as the number of packages containing three products. The probability that a product is faulty is $1/10$.
The production engineer randomly selects products for quality assurance.
Let X be the number of products in a package and Y the number of faulty products in a package.

a. What is the probability distribution of Y, given that $X=3$?
b. What is the probability distribution of Y, given that X is any K?
c. What is the expectation of the number of faulty products in a package of three products? Explain.
d. Construct the joint probability function.

4) A container has three balls numbered 2, 4, and 8. Two balls are removed **without returns**. We define X as the smallest number on the balls and Y as the largest number.

a. Calculate the probability distribution of (X,Y).
b. If the lowest number selected is 2, what are the chances that the largest number is 8?
c. Calculate the dependent probability distribution of X, given that $Y=4$. Find: $\mathrm{E}(X \mid Y=4)$.

5) There are two bank branches in a community: City Bank and Union Bank.
The following data applies to the adult population in the community:
60% have an account in City Bank.
50% have an account in Union Bank.
95% have an account in at least one branch.
Let X be the number of branches in which an adult resident of the community has an account. Let Y be an indicator variable, such as $Y=1$ if the person has an account at City Bank, $Y=0$ otherwise.

a. Construct the joint probability function of X and Y.
b. Add the marginal probability function.
c. It is known that a given resident has an account at City Bank. What is the probability that he has no other account?

Answer Key

1) a.

Y \ X	0	50	100	$P(Y)$
1	0	0	0.3	0.3
3	0.343	0.294	0.063	0.7
$P(X)$	0.343	0.294	0.363	1.0

b. $E(Y) = 2.4$

c. $E(Y) = 1.347$, $V(Y) = 0.575$

2) a.

Y \ X	0	1	2	$P(Y)$
2	0.2	0.08	0.12	0.4
3	0.1	0.05	0	0.15
4	0.1	0.27	0.08	0.45
$P(X)$	0.4	0.4	0.2	1.0

b. Yes.

c. 0.125

3) a. $Y|_{X=3} \sim B\left(n = 3, p = \frac{1}{10}\right)$

b. $Y|_{X=k} \sim B\left(n = k, p = \frac{1}{10}\right)$

c. $E\left(Y|_{X=3}\right) = 0.3$

d.

Y \ X	2	3	$P(Y)$
0	0.405	0.3645	0.7695
1	0.09	0.1215	0.1224
2	0.005	0.0135	0.014
3	0	0.0005	0.0005
$P(X)$	0.5	0.5	1

4) a.

Y \ X	2	4	$P(Y)$
4	$\frac{1}{3}$	0	$\frac{1}{3}$
8	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$
$P(X)$	$\frac{2}{3}$	$\frac{1}{3}$	1

b. $\frac{1}{2}$

c. $E(X|Y = 4) = 2$

5) a.+b.

Y \ X	0	1	2	$P(Y)$
0	0.05	0.35	0	0.4
1	0	0.45	0.15	0.6
$P(X)$	0.05	0.8	0.15	1

c. $E(X = 1|Y = 1) = 0.75$

Correlation between Variables

Questions

1) The chances of a student passing the statistics exam on the first try are 0.8.
If he fails on the first try, he takes the exam again, on which his chances of passing are 0.9 (a student who passed on the first try does not take another exam).
If the student fails on the second try, he files a request to take the exam for the third time.
The chances of the request being approved are 0.2, and the chances of actually passing it are 0.7.
We define X as the number of exams taken by the student.
We define Y as the number of exams on which he failed.

a. Construct the joint probability function of X and Y, and their marginal probability functions.
b. Are X and Y independent variables?
c. It is known that the student took more than one exam.
What is the probability that he failed less than three exams?
d. Are X and Y correlated completely or partially? Positively or negatively?
Explain without any calculations.
e. Calculate the correlation coefficient of X and Y.
f. Are the variables uncorrelated?

2) A coin is tossed three times. We define X as the number of tails obtained on the first two tosses and Y as the number of tails obtained on the last two.

a. Construct the joint probability function of X and Y, and their marginal probability functions.
b. Are X and Y independent variables?
c. What is the correlation coefficient of X and Y?
Are they correlated variables?
d. Assume that exactly one of the first two tosses was tails.
What is the probability that the last two tosses were tails?
e. If there were at least one tails on the last two tosses, what is the probability of exactly one tails on the first two tosses?

3) Three different balls are distributed among three different boxes.
We define the following variables:
X is the number of balls in the first box.
Y is the number of balls in the second box.

a. Construct the joint probability function of X and Y.
b. Are the variables uncorrelated?

4) A balanced die is thrown twice.
X is the larger of the two throws.
Y is the number of throws with odd numbers.

a. Find the joint probability function of X and Y.
b. Calculate the correlation coefficient of X and Y.

5) Find the probability distribution of Y, given that $X = 2$.
A building has five apartments.
Apartment $\#1$ and Apartment $\#3$ have been renovated, while the others have not.
It is decided to randomly select two apartments from the building.
We define the following variables:
X is the number of renovated apartments selected.
Y is the number of odd-numbered apartments selected.

a. Construct the joint probability function and the marginal probability functions.
b. Are the variables correlated?
c. What is the correlation coefficient of X and Y?
d. What is the correlation coefficient:
    i. Between the number of renovated apartments sampled and the number of even-numbered apartments sampled?
    ii. Between the number of even-numbered apartments and the number of odd-numbered apartments sampled?
e. Every renovated apartment cost $\$2$ million and every un-renovated apartment cost $\$1.5$ million.
What is the correlation between the cost of the sampled apartments and the number of even-numbered apartments?

Answer Key

1)

Y \ X	1	2	3	P(Y)
0	0.8	0	0	0.8
1	0	0.18	0	0.18
2	0	0.016	0.0028	0.0188
3	0	0	0.0012	0.0012
P(X)	0.8	0.196	0.004	1

b. Yes.
c. 0.994
d. Partially and Positively correlated.
e. 0.963
f. They are correlated.

2)

Y \ X	0	1	2	P(Y)
0	$\frac{1}{8}$	$\frac{1}{8}$	0	$\frac{2}{8}$
1	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	$\frac{4}{8}$
2	0	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{2}{8}$
P(X)	$\frac{2}{8}$	$\frac{4}{8}$	$\frac{2}{8}$	1

b. Dependent.
c. Correlation coefficient:
0.5 – correlated.
d. 0.25
e. 0.5

3)

Y \ X	0	1	2	3	P(Y)
0	$\frac{1}{27}$	$\frac{3}{27}$	$\frac{3}{27}$	$\frac{1}{27}$	$\frac{8}{27}$
1	$\frac{3}{27}$	$\frac{6}{27}$	$\frac{3}{27}$	0	$\frac{12}{27}$
2	$\frac{3}{27}$	$\frac{3}{27}$	0	0	$\frac{6}{27}$
3	$\frac{1}{27}$	0	0	0	$\frac{1}{27}$
P(X)	$\frac{8}{27}$	$\frac{12}{27}$	$\frac{6}{27}$	$\frac{1}{27}$	1

b. Correlated.

4) a.

Y \ X	1	2	3	4	5	6	$P(Y)$
0	$\frac{1}{36}$	0	$\frac{3}{36}$	0	$\frac{5}{36}$	0	$\frac{9}{36}$
1	0	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{4}{36}$	$\frac{4}{36}$	$\frac{6}{36}$	$\frac{18}{36}$
2	0	$\frac{1}{36}$	0	$\frac{3}{36}$	0	$\frac{5}{36}$	$\frac{9}{36}$
$P(X)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$	1

b. 0.252

c.

$Y \mid x=2$	1	2
$P(Y \mid x=2)$	$\frac{1}{3}$	$\frac{2}{3}$

5) a.

X \ Y	0	1	2	$P(Y)$
0	0.1	0	0	0.1
1	0.2	0.4	0	0.6
2	0	0.2	0.1	0.3
$P(X)$	0.3	0.6	0.1	1

b. Yes, they are. c. $\frac{2}{3}$ d. i. $\frac{2}{3}$ ii. -1 e. $-\frac{2}{3}$

The Two-Dimensional Random Variable - Linear Combinations

Questions

1) Given the following joint probability functions:

a. Fill in the missing probabilities
b. Are the variables dependent?
c. Are the variables uncorrelated?
d. Calculation the covariance.
e. Calculate the expectation and variance of the sum of the two variables.
f. Calculate the expectation and variance of the difference of the two variables

Y \ X	1	2	3	$P(Y)$
2		0.1	0.3	0.6
3	.02		.01	
$P(X)$				1

2) An exam is composed of a quantitative part and a verbal part.
The expectation of the marks in the quantitative part is 100 with a standard deviation of 20. The expectation of the marks in the verbal part is 90 with a standard deviation of 15.
The correlation coefficient of the marks on the quantitative and verbal parts is 0.8.

a. Calculate the covariance of the quantitative mark and the verbal mark.
b. Calculate the expectation and variance of the sum of the marks on the quantitative part and the verbal part.
c. Calculate the expectation and variance of the difference between the marks on the quantitative part and the verbal part.
d. The exam costs $\$2,000$. It is decided to reimburse $\$1$ for each point accumulated on the verbal part and $\$2$ for each point accumulated on the quantitative part. What are the expectation and variance of the net cost of the exam (cost minus reimbursement)?

3) Assume that $\text{Var}(X-2Y)=2$ and $\text{Var}(X+2Y)=3$.
Calculate $\text{Cov}(X,Y)$.

4) A die is thrown n times.
We define the following variables:
X is the number of times 6 is thrown.
Y is the number of times 5 is thrown.
Express the covariance of X and Y in terms of n.

Answer Key

1) a.

Y \ X	1	2	3	$P(Y)$
2	0.2	0.1	0.3	0.6
3	0.2	0.1	0.1	0.4
$P(X)$	0.4	0.2	0.4	1

b. Yes. c. Correlated. d. -1

e. $E(X+Y)=4.4$, $V(X+Y)=0.84$ f. $E(X-Y)=-0.4$, $V(X-Y)=1.24$

2) a. $Cov(X,Y)=240$ b. $E(X+Y)=190$, $V(X+Y)=1105$

c. $E(X-Y)=10$, $V(X-Y)=145$ d. $E(W)=1710$, $V(W)=2785$

3) $Cov(X,Y)=0.125$

4) $Cov(X,Y)=\frac{-n}{36}$

Discrete Two-Dimensional Random Variable - Summary Questions

Questions

1) You are required to create a password using 3 characters from the following list of characters: $A, B, C, 1$ and 2.
The following variables are defined:
X - the number of times that the character "1" appears in the password.
Y - the number of times that the number "1" appears in the first and/or last place in the password.

a. Identify the marginal distributions of X and Y
b. Find the joint distribution of X and Y
c. Find the correlation coefficient of X and Y
d. What is the correlation coefficient between $2X$ and $3Y+5$?

2) At a graduation party there is an ice filled tub containing 7 bottles of beer:
4 Coors, 2 Budweiser's and 1 Miller.
Jane took 3 bottles of beer at random from the tub.
The following variables are defined:
X - the number of Coors bottles taken from the tub by Jane;
Y - the number of Miller bottles taken from the tub by Jane.

a. Construct the joint distribution of X and Y
b. Calculate the expectation and variance of X and Y
c. Find the covariance of X and Y.
d. Define W as the number of Budweiser bottles taken by Jane.
Express W in terms of X and Y, and calculate the expectation and variance of W based on the results from only sections b and c above.
e. What is the correlation coefficient between the number of Coors bottles and the number of non-Coors bottles that were taken by Jane?

3) There are 6 pairs of shoes in a drawer.
Jack removed 4 individual shoes at random from the drawer.
The following variables are defined:
W - the number of pairs of shoes that Jack removed.
R - the number of left shoes that Jack removed.

a. Find the joint distribution of W and R.
b. Are W and R independent variables?
c. Find the distribution of R (the number of left shoes that were removed) given that in total only one pair of shoes were removed.
d. Given that at least 3 left shoes were removed, what are the chances that at most one pair of shoes was removed?

4) A jar contains 5 blue balls, 4 white balls and 3 green balls.
3 balls are randomly chosen, one at a time without returns.
The following variables are defined:
X – receiving a value of 1 if at least one blue ball is chosen, 0 otherwise
Y – the number of white balls that were chosen.

a. Calculate $\mathrm{P}(X=1)$.
b. Build the joint probability distribution of X and Y.
c. What is the expectation of Y if no blue balls were removed?
d. What is the variance of X if it's known that at least one white ball was removed?

5) On his 4^{th} birthday, Johnny gave out 3 different presents that were chosen at random to 5 children who were also chosen at random.
Each time that Johnny gave out a present, he randomly chose a child independently of his previous choices.
The following variables are defined:
X – the number of presents given to Jane;
Y – the number of children who did not receive any presents.

a. Build the joint probability distribution of X and Y.
b. Are X and Y uncorrelated?
c. Calculate: $\mathrm{E}\left(X \cdot Y^2\right)$.
d. What is the correlation coefficient between X (the number of presents given to Jane) and the number of children that received a present?

6) Which of the sentences below are true?

a. If two variables are correlated, then they are dependent.
b. If two variables are dependent, then they are correlated.
c. If two variables are independent, then they are uncorrelated.
d. If two variables are uncorrelated, then they are independent.

7) There are 50 employees working at the ACME factory - 25 men and 25 women.
Each employee is asked to choose a present for Christmas.
The employees have 5 different presents to choose from, and must choose only one present.
Each employee randomly chooses one present, independent of other employees.
The following variables are defined:
X_i – the number of men who choose present i .
Y_i – the number of women who choose present i.

a. Are X_1 and Y_1 independent variables? Explain.
(there is no need for calculations)
b. Are X_1 and X_2 independent variables? Explain.
(there is no need for calculations)
c. What is the distribution of $X_1 + X_2$?
d. Is there a full or partial correlation between X_1 and X_2? Positive or Negative?
Explain (there is no need for calculations).

8) Prove the following equation for X, Y and Z: $\text{cov}(X+Y,Z)=\text{cov}(X,Y)+\text{cov}(Y,Z)$.

9) In autumn, the number of leaves falling at a garden has a Poisson distribution with and expectation of 50 leaves per minute.
The following variables are defined:
Y – the number of leaves falling between $12{:}00$ and $12{:}10$.
Q – the number of leaves falling between $12{:}05$ and $12{:}30$.

a. Calculate $\text{cov}(4Y,Q+6)$.
b. What is the correlation coefficient between Y and Q?

10) A basket contains 20 red balls, 20 green balls and 20 blue balls.
20 balls are randomly chosen.
Calculate the correlation coefficient between the number of red balls and the number of green balls that were chosen.

11) Given $Y \sim B(n=1,p)$ where $0<p<1$.
Prove that if $P(X=x \mid Y=1)$ for all X, then X and Y are independent variables.

12) Given two independent random variables X, Y where:
$X \sim B(n,p)$
$Y \sim B(m,p)$
Prove that: $X \mid X+Y=k \sim HG(n+m,n,k)$.

Answer Key

1)

a.

$X \sim B\left(n=3, p=\frac{1}{5}\right)$

$Y \sim B\left(n=2, p=\frac{1}{5}\right)$

c. $Cov(X,Y)=0.816$

d. $\rho_{2x,3y+5}=\rho_{xy}=0.816$

b.

Y \ X	0	1	2	3	$P(Y)$
0	$\frac{64}{125}$	$\frac{16}{125}$	0	0	$\frac{80}{125}$
1	0	$\frac{32}{125}$	$\frac{8}{125}$	0	$\frac{40}{125}$
2	0	0	$\frac{4}{125}$	$\frac{1}{125}$	$\frac{5}{125}$
$P(X)$	$\frac{64}{125}$	$\frac{48}{125}$	$\frac{12}{125}$	$\frac{1}{125}$	1

2)

a.

Y \ X	0	1	2	3	$P(Y)$
0	0	$\frac{4}{35}$	$\frac{12}{35}$	$\frac{4}{35}$	$\frac{20}{35}$
1	$\frac{1}{35}$	$\frac{8}{35}$	$\frac{6}{35}$	0	$\frac{15}{35}$
$P(X)$	$\frac{1}{35}$	$\frac{12}{35}$	$\frac{18}{35}$	$\frac{4}{35}$	1

b. $E(X)=\frac{12}{7}$, $V(X)=\frac{24}{49}$

$E(Y)=\frac{3}{7}$, $V(Y)=\frac{12}{49}$

c. $Cov(X,Y)=-\frac{8}{49}$

d. $E(W)=\frac{6}{7}$; $V(W)=\frac{20}{49}$

e. $\rho_{X,R}=-1$

3)

a.

R / W	0	1	2	$P(R)$
0	$\frac{15}{495}$	0	0	$\frac{15}{495}$
1	$\frac{60}{495}$	$\frac{60}{495}$	0	$\frac{120}{495}$
2	$\frac{90}{495}$	$\frac{120}{495}$	$\frac{15}{495}$	$\frac{225}{495}$
3	$\frac{60}{495}$	$\frac{60}{495}$	0	$\frac{120}{495}$
4	$\frac{15}{495}$	0	0	$\frac{15}{495}$
$P(W)$	$\frac{240}{495}$	$\frac{240}{495}$	$\frac{15}{495}$	1

b. The variables are dependent

c. 1

4)

a. $P(X=1)=\frac{185}{220}$

b.

X / Y	0	1	$P(Y)$
0	$\frac{1}{220}$	$\frac{55}{220}$	$\frac{56}{220}$
1	$\frac{12}{220}$	$\frac{100}{220}$	$\frac{112}{220}$
2	$\frac{18}{220}$	$\frac{30}{220}$	$\frac{48}{220}$
3	$\frac{4}{220}$	0	$\frac{4}{220}$
$P(X)$	$\frac{35}{220}$	$\frac{185}{220}$	1

c. $E(Y|x=0)=1.714$

d. $V(X|y\geq 1)=0.164$

5)

a.

X / Y	0	1	2	3	$P(Y)$
0	$\frac{24}{125}$	$\frac{36}{125}$	0	0	$\frac{60}{125}$
1	$\frac{36}{125}$	$\frac{12}{125}$	$\frac{12}{125}$	0	$\frac{60}{125}$
2	$\frac{4}{125}$	0	0	$\frac{1}{125}$	$\frac{5}{125}$
$P(X)$	$\frac{64}{125}$	$\frac{48}{125}$	$\frac{12}{125}$	$\frac{1}{125}$	1

b. X and Y are uncorrelated.

c. $E(X \cdot Y^2) = 4.128$

d. 0

6) a. True b. False c. True d. False

7) a. Independent b. Dependent

c. $X_1 + X_2 \sim B\left(n = 25, p = \frac{2}{5}\right)$ d. Partial and negative.

8) $Cov(X+Y,Z) = E[(X+Y)\cdot Z] - E(X+Y)\cdot E(Z)$

$= E(XZ+YZ) - E(X+Y)\cdot E(Z) = E(XZ) + E(YZ) - [(E(X)+E(Y))E(Z)]$

$= E(XZ) + E(YZ) - E(X)\cdot E(Z) + E(Y)\cdot E(Z) = E(XZ) - E(X)\cdot E(Z) + E(YZ)\cdot E(Z)$

Which is: $\text{cov}(X,Y) + \text{cov}(Y,Z)$.

9) a. 1000 b. 0.316

10) $-\frac{1}{2}$

11) Solution in the recording.

12) Solution in the recording.