proprep™

Workbook

Properties of Analytic Functions

Questions

Liouville's Theorem

1) Find an entire function *f* which satisfies the inequality $\left|\sin z - zf(z)\right| < 2$ for all $z \in \mathbb{C}$.
Hint: use Liouville's Theorem.

2) Find an entire function *f* which satisfies $\left|\sin^2 z \cdot \cos z - z^2 \cdot f(z)\right| < 100$ for all $z \in \mathbb{C}$.

3) Find an entire function *f* which satisfies $\left|z\cos z + \left(z - \frac{\pi}{2}\right) f(z)\right| < 200$ for all $z \in \mathbb{C}$.

4) Find all entire functions $f(z) = u + iv$ such that $u \le 0$ for all *z*.
Hint: consider the function $e^{f(z)}$.

5) Find all entire functions $f(z) = u + iv$ such that $u \ge 0$ for all *z*.
Hint: consider the function $e^{-f(z)}$.

6) Find all entire functions $f(z) = u + iv$ such that $v \ge 0$ for all *z*.
Hint: consider the function $e^{i \cdot f(z)}$.

7) Let $f(z)$ be an entire functions such that $f(z) = u + iv$ such that $\left|f(z)\right| \ge 1$ for all *z*.
Prove that *f* is constant.
Hint: consider the function $\frac{1}{f(z)}$.

8) Find all entire functions $f(z)=u+iv$ such that $v\le 0$ for all z.

Hint: consider the function $e^{-i\cdot f(z)}$.

9) Prove that if $f(z)$ is entire and satisfies $\left|f(z)\right|\le 4+5|z|^{\frac{4}{5}}$ for all z then f is constant.

10) Prove that if $f(z)$ is entire and satisfies $\left|f(z)\right|\ge e^{\operatorname{Re}(z)}$ $\forall z\in\mathbb{C}$ then there is a constant $c\in\mathbb{C}$ such that $f(z)=ce^{z}$ $\forall z\in\mathbb{C}$.

11) Prove that if $f(z)$ is entire and satisfies $f(0)=0$ and $f(1)=1$ then there is a constant $c\in\mathbb{C}$ such that $f(c)\ge 2$.

12) Prove that if $f(z)$ is entire and satisfies $\lim\limits_{z\to\infty} f(z)=2$ then $f(z)\equiv 2$.

13) Prove that if $f(z)=u+iv$ is entire and satisfies $u\cdot v\ge 0$ $\forall z\in\mathbb{C}$ then f is constant.

Hint: consider $g(z)=f^{2}(z)$ and then $h(z)=e^{ig(z)}$.

14) Prove that if $f(z)=u+iv$ is entire and satisfies $u\ge v$ $\forall z\in\mathbb{C}$ then f is constant.

Hint: consider $g(z)=f(z)+i\cdot f(z)$ and then $h(z)=e^{-g(z)}$.

15) Prove no analytic function $f(z)$ in $\mathbb{C}^{\times}=\mathbb{C}-\{0\}$ can satisfy $|f(z)|\ge\frac{1}{\sqrt{|z|}}\forall z\in\mathbb{C}^{\times}$. You may use Riemann's Extension Theorem: If $D\subseteq\mathbb{C}$ be is a domain containing z_0 and if $g(z)$ is analytic and bounded on $D-\{z_0\}$, then g extends uniquely to an analytic function $\tilde{g}$ on all of D.

16) Prove that if $f(z)$ is entire and $\forall z\in\mathbb{C}$, $\operatorname{Re} f(z)\le\left|f(z)\right|^{2}$ then $f(z)\equiv const$

17) Prove that if $f(z)$ is entire and not constant then the image $f(\mathbb{C})$ is dense in $\mathbb{C}$. Definition: a set $A\subseteq\mathbb{C}$ is called dense in $\mathbb{C}$ if $\forall z_0\in\mathbb{C}$ and $\forall\varepsilon>0$ $D(z_0,\varepsilon)\cap A\neq\emptyset$.

18) It is known that there exists an function $T:\mathbb{C}-(-\infty,0]\to D(0,1)$ satisfying

$T'(z)\neq 0\,\forall z$. [For example $T(z)=e^{-\sqrt{z}}$ where $\sqrt{z}\equiv e^{\frac{1}{2}\text{Log}(z)}$].

Suppose that $f:\mathbb{C}\to\mathbb{C}-(-\infty,0]$ is an entire function. Prove that f is constant

The Identity Theorem

19) Prove that if $f(z)$ is entire and satisfies $f\left(\frac{1}{n}\right)=\frac{1}{n}\ \forall n\in\mathbb{N}$, then $f(z)\equiv z$.

$\mathbb{N}=\{1,2,3,\dots\}$

20) Suppose $f(z)$ is analytic on $D=D(0,1)$ and satisfies $f\left(\frac{1}{n}\right)=\frac{1}{n+1}\ \forall n\in\mathbb{N}$.

Find $f(z)$.

21) Suppose $f(z)$ is analytic on $D=D(0,1.5)$ and satisfies

$f\left(\frac{1}{2n}\right)=\frac{1}{2\pi i}\oint_{|z|=1}\frac{nz}{nz-0.5}dz\ \forall n\in\mathbb{N}$. Find $f(z)$.

22) Suppose $f(z)$ is analytic on $D=D(0,1)$ and satisfies $f\left(\frac{1}{3n-1}\right)=\frac{2}{n}\ \forall n\in\mathbb{N}$

. Find $f(z)$.

23) Prove that if $f(z)$ is analytic on $D=D(0,1)$ and satisfies

$f\left(\frac{1}{n}\right)=\sin(\pi n)\ \forall n\in\mathbb{N}$ then $f(z)\equiv 0$ in D.

24) Find all analytic functions $f(z)$ on $D=D(0,1)$ which satisfy

$$f\left(\frac{1}{n}\right)=\begin{cases}\frac{1}{n+1}\ ;\ n=2k & (n\text{ even})\\ \frac{1}{n+2}\ ;\ n=2k-1 & (n\text{ odd})\end{cases}$$

25) Find an analytic function $f(z)$ on $D=D(0,1)$ which has infinitely many zeros in D,

or prove than no such f exists. Recall: $D(0,1)=\{z\,|\,|z|<1\}$ (open unit disk).

26) Is there an analytic function $f(z)$ on $D=\{z\,|\,1<|z|<3\}$ such that $f(x)=|x|^3$ for all

real x such that $1<|x|<3$? Find such an f or prove than no such f exists.

27)

a) Prove the following theorem:

If $f(z)$, $g(z)$ are analytic on a domain D and $f(z)g(z)\equiv 0$, then $f(z)\equiv 0$ or $g(z)\equiv 0$.

b) Let $f(z), g(z)$ be analytic in a domain D and let $a,b\in\mathbb{C}$ (arbitrary).

Prove that if $\left(f(z)-a\right)\left(g(z)-b\right)\equiv 0\ \forall z\in D$ then $f(z)\equiv a$ or $g(z)\equiv b$.

28) Note: This exercise assumes knowledge of the Residue Theorem.

Suppose that $f(z)$ is analytic on $D=D(z_0,R)$, where $z_0\in\mathbb{C}, R>0$; and that $f'(z_0)\neq 0$. Prove that $\exists r$ such that $0<r<R$

$$\frac{2\pi i}{f'(z_0)}=\oint_{|z-z_0|=r}\frac{1}{f(z)-f(z_0)}dz$$

29) Note: This exercises assumes knowledge of Poles [a kind of Isolated Singular Point].

Define the domain $D=\{z\,|\,0<|z|<1\}$, the punctured (open) unit disk.

Suppose that: $f(z)$ is analytic in D, has a pole at $z=0$, and

$$f\left(\frac{1}{n}\right)=n\ \ \forall n\in\mathrm{N}$$

Prove that $f(z)=\dfrac{1}{z}\ \forall z\in D$

30) Given an analytic function $f(z)$ on $|z|>1$ such that for all real $x>1$, $f(x)$ is real.

Prove that, for all real $x<-1$, $f(x)$ is real.

31) Given a continuous function $f(z)$ on $D = D(0,1)$ which satisfies

$$f\left(\frac{1}{n}\right) = \frac{1}{n^2} \ \forall n \in \mathrm{N} \quad \text{and} \quad f\left(\frac{i}{2}\right) = 0 .$$

Prove that f is not analytic on D.

32)

$f(z)$ is analytic on $D = D(0,1)$, continuous on $\overline{D}$, and satisfies $|f(z)| = 1 \forall z \in \partial D$.

Notation: $D : |z| < 1,\ \overline{D} : |z| \le 1,\ \partial D = 1$

Prove that f can have only finitely many zeros in D.

Hint: use the Bolzano-Weierstrass theorem:

Every bounded sequence has a convergent subsequence.

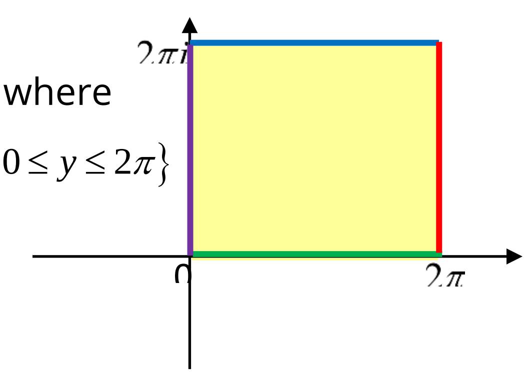
The Maximum and Minimum Modulus Principles

33) Let $f(z) = e^{-z^2}$ in $D = D(0,1)$. Does $|f|$ have a maximum value in D? If so, find it

34) Let $f(z) = e^{-z^2}$ and let $D = D(0,3)$. Does f have a maximum value in $\overline{D}$? If so, find it.

35) True or False? If $f(z)$ is bounded on the domain $|z| > 1$, it must be constant.

36) Find the maximum value (if it exists) of $|f(z)|$, where $f(z) = \cos z$, on the set $K = \{z = x + iy \,|\, 0 \le x \le 2\pi,\ 0 \le y \le 2\pi\}$



37) Let $D = D(0,1)$ and let $f(z) = e^{z^2}$ in $\bar{D}$. Does f have a maximum value in $\bar{D}$? If so, find it and where it occurs.

38) Let f be analytic on $|z| < R$ and continuous on $K: |z| \le R$ such that $|f(z)| > a$ on $\partial K: |z| = R$ and $|f(0)| < a$, for some $a > 0$. Prove that f has at least one zero in $|z| < R$

39) Prove:

a) If $f(z)$ is a nonconstant analytic function on an open set *D* and $f(z) \ne 0$, then $|f(z)|$ can't have a local minimum in *D*.

b) Let $f(z)$ be an analytic function on an open set *D* such that $f(z) \ne 0$ If $|f(z)|$ has a local minimum in *D*, then $f(z)$ is a constant function.

c) If $f(z)$ is analytic on a bounded open set *D*, and is nonzero and continuous on $\bar{D}$, then $|f(z)|$ achieves its minimum on ∂D.

40) Let $f(z)$ be analytic and nonzero on $K: |z| \le 1$ and suppose that $|f(z)| = 1$ on $\partial K: |z| = 1$.

a) Prove that $f(z)$ is constant.

b) Is the conclusion still true if we drop the requirement that $f(z)$ be nonzero?

41) Let $f(z) = u + iv$ be analytic on a bounded open set *D*, and continuous on $\bar{D}$ Prove that $u(x, y)$ attains its maximum on ∂D.

42) Let $f(z) = u + iv$ be analytic on a bounded open set *D*, and continuous on $\bar{D}$ Prove that $u(x, y)$ attains its minimum on ∂D.

43) Let $f(z)$, $g(z)$ be analytic on $D:|z|<1$, and continuous on $\bar{D}:|z|\le 1$, for all z in $\partial D: |z| = 1$ we have $\mathrm{Re}\{f(z)\}=\mathrm{Re}\{g(z)\}$. Prove: $f(z)\equiv g(z)+i\cdot c$ on $\bar{D}$ where c is a real constant.

44) Let $p(z)=\sum_{k=0}^{n} a_k z^k$ be a polynomial of degree n $(a_n \ne 0)$ such that $|p(z)|=1$ on $|z|=1$. Prove that $|p(z)|\le |z|^n$ on $|z|\ge 1$.

Hint: show that $f(z)=z^n p\left(\frac{1}{z}\right)$ on $\mathbb{C}^\times$ can be extended to an entire function.

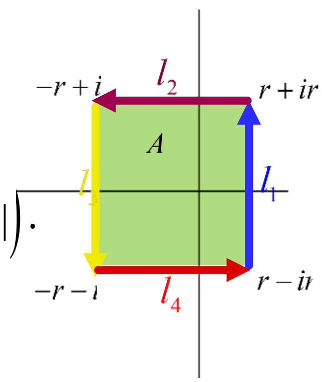
45) Let $f(z)$ be continuous on the compact set

$A=\{z=x+iy \mid -r\le x\le r, -r\le y\le r\}$ $(r>0)$,

and analytic on A°. We can write $\partial A = l_1\cup l_2\cup l_3\cup l_4$, where [see picture]

$l_1=\{r+iy \mid -r\le y\le r\}$ $\quad$ $l_3=\{-r+iy \mid -r\le y\le r\}$

$l_2=\{x+ir \mid -r\le x\le r\}$ $\quad$ $l_4=\{x-ir \mid -r\le x\le r\}$



Prove that $|f(0)|\le \frac{1}{4}\left(\max_{z\in l_1}|f(z)|+\max_{z\in l_2}|f(z)|+\max_{z\in l_3}|f(z)|+\max_{z\in l_4}|f(z)|\right)$.

Hint: $g(z)=\frac{1}{4}[f(z)+f(-z)+f(iz)+f(-iz)]$

46) Prove that if $A\subseteq\mathbb{C}$ is open and $f(z)$ is analytic and nonconstant on A then $f(A)$ is open.

47) Let $D=D(0,1)=\{z\in\mathbb{C} \mid |z|<1\}$ and let f be a function such that:

a) f is analytic on D

b) $|f(z)|\le 1$ for all $z\in D$

c) $f(0)=0$

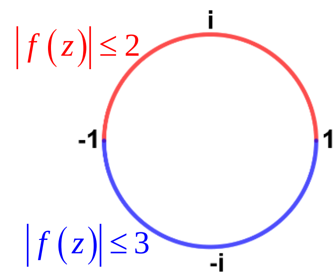
Prove that $|f(z)|\le |z|\ \forall z\in D$ and that $|f'(0)|\le 1$.

48) Suppose that $f(z)$ is analytic on $\overline{D(0,1)}=\{z\in\mathbb{C} \mid |z|\le 1\}$ and $|f(z)|=1$ when $|z|=1$. Suppose, in addition, that $f(z)=0 \Leftrightarrow z=0$.

Prove that $f(z)\equiv c\cdot z^k$ for some $c\in\mathbb{C}$ and $k\in\mathrm{N}$.

49) Suppose $f(z)$ is analytic on the annulus $A=\{z\in\mathbb{C} : 1<|z|<2\}$ and continuous on $\bar{A}$. Suppose, in addition, that $|f(z)|=1$ on $|z|=1$ and that $|f(z)|=8$ on $|z|=2$. Prove that $|f(z)|\le|z|^3$ for all $z\in A$.

50) Suppose $f(z)$ is analytic on $|z|\le 1$ and satisfies:



a) $|f(z)|\le 2$ on $\{z \mid \operatorname{Im} z\ge 0 \text{ and } |z|=1\}$

b) $|f(z)|\le 3$ on $\{z \mid \operatorname{Im} z\le 0 \text{ and } |z|=1\}$

Prove that $|f(0)|\le\sqrt{6}$.

Hint: consider $g(z)=f(z)\cdot f(-z)$.

51) Let $f(z)$ be analytic and nonzero $D=D(0,1)$. Prove that there exists a sequence $\{z_n\}\subseteq D$ such that: $|z_n|\to 1$ and $\{f(z_n)\}$ is bounded.

Answer Key

1) $f(z)=\begin{cases}\dfrac{\sin z}{z}; & z\neq 0\\ 1\quad ; & z=0\end{cases}$

2) $f(z)=\begin{cases}\left[\dfrac{\sin z}{z}\right]^2\cos z; & z\neq 0\\ 1\quad ; & z=0\end{cases}$

3) $f(z)=\begin{cases}\dfrac{z\cos z}{\frac{\pi}{2}-z}; & z\neq \dfrac{\pi}{2}\\ \dfrac{\pi}{2}\quad ; & z=\dfrac{\pi}{2}\end{cases}$

4) $f(z)\equiv a+bi,\ a\leq 0$

5) $f(z)=a+bi,\ a\geq 0$

6) $f(z)=a+bi,\ b\geq 0$

7) Proof

8) $f(z)=a+bi,\ b\leq 0$

9) Proof

10) Proof

11) Proof

12) Proof

13) Proof

14) Proof

15) Proof

16) Proof

17) Proof

18) Proof

19) Proof

20) Proof

21) $f(z) = \dfrac{z}{1+z}$

22) $f(z) = z$

23) $f(z) = 6\dfrac{z}{z+1}$

24) Proof

25) No such f exists.

26) $f(z) = sin\left(\frac{\pi}{1-z}\right)$, for example.

27) No such f exists.

28) Proof

29) Proof

30) Proof

31) Proof

32) Proof

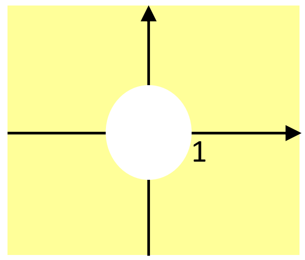
33) Proof

34) No maximum.

35) e^9

36) False.

Example, $f(z)=\dfrac{1}{z}$.



37) $\cosh 2\pi \approx 268$

38) Maximum is *e*. Occurs at $z=\pm 1$.

39) Proof

40) Proof

41)

a) Proof

b) No

42) Proof

43) Proof

44) Proof

45) Proof

46) Proof

47) Proof

48) Proof

49) Proof

50) Proof

51) Proof

52) Proof