proprep

Workbook

Table of Contents

Fourier Series

Real Fourier Series

Questions

1) Compute the real Fourier series for the function $f(x)$ on the interval $[-\pi,\pi]$,

where $f(x)=\begin{cases}1 & 0<x<\pi \\ 0 & -\pi<x<0\end{cases}$.

2) Compute the real Fourier series for the function $f(x)$ on the interval $[-\pi,\pi]$,

where $f(x)=\sin|x|$.

3) Compute the real Fourier series for the function $f(x)$ on the interval $[-\pi,\pi]$,

where $f(x)=\begin{cases}1 & -1<x<1 \\ 0 & \text{otherwise}\end{cases}$.

Answer Key

To view the answers to those exercises, please refer to the appropriate videos on site.

Complex Fourier Series

Questions

1) Compute the complex Fourier series for the function $f(x)$ on the interval $[-\pi,\pi]$,

where $f(x)=\begin{cases} 0 & 0<x<\pi \\ -x & -\pi<x<0 \end{cases}$.

2) Compute the complex Fourier series for the function $f(x)$ on the interval $[-\pi,\pi]$,

where $f(x)=\begin{cases} x & -\pi<x<0 \\ 2x & 0<x<\pi \end{cases}$.

3) Compute the complex Fourier series for the function $f(x)$ on the interval $[-\pi,\pi]$,

where $f(x)=\begin{cases} 1 & -\pi<x<0 \\ -2 & 0<x<\pi \end{cases}$.

Answer Key

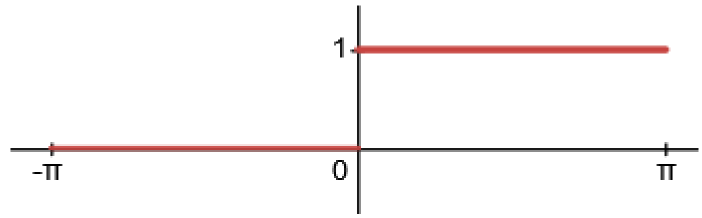
To view the answers to those exercises, please refer to the appropriate videos on site.

Parseval Identity

Questions

1) Let $f(x)=\begin{cases}1 & 0<x<\pi\\ 0 & -\pi<x<0\end{cases}$ on $[-\pi,\pi]$. Given that $f\sim\frac{1}{2}+\sum_{k=1}^{\infty}\frac{2}{\pi(2k-1)}\sin((2k-1)x)$,

prove that $\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}=\frac{\pi^2}{8}$.



2) a. Compute the real Fourier series of the function $f(x)=\sin\frac{px}{2}$ $(p\in\mathbb{R})$ on $[-\pi,\pi]$.

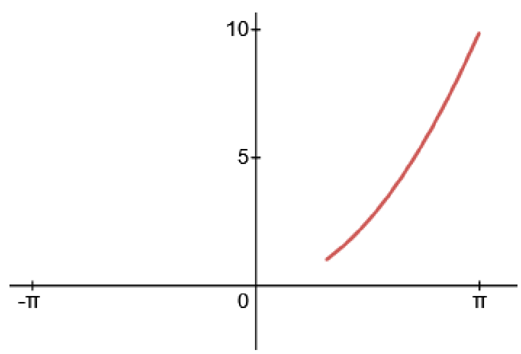
Assume p not an even number ($p\neq 2n\ \forall n\in\mathbb{Z}$).

b. Prove the identity $\sum_{n=1}^{\infty}\frac{n^2}{(1-4n^2)^2}=\frac{\pi^2}{64}$.

3) a. Compute the real Fourier series of the function $f(x)=\begin{cases}h^2 & h<x<\pi\\ 0 & -\pi<x<h\end{cases}$ on $[-\pi,\pi]$,

where $0\neq h\in[-\pi,\pi]$.

b. Compute $\sum_{n=1}^{\infty}\frac{1-(-1)^n\cos(2n)}{n^2}$.



4) Let f be a 2π-periodic function such that $f(x)=\sinh x=\frac{e^x-e^{-x}}{2}$ for $-\pi<x<\pi$.

a. Sketch the graph of f on the interval $-3\pi<x<3\pi$.

b. Find the real Fourier series of f.

c. Compute the sum of the series $\sum_{n=1}^{\infty}\frac{n^2}{(1+n^2)^2}$.

5) Define f on $[-\pi,\pi]$ by $f(x)=\sum_{n=1}^{\infty}\sqrt{\frac{1}{n^2}-\frac{1}{(n+2)^2}}\cdot e^{inx}$. Compute $\int_{-\pi}^{\pi}\left|f(x+\pi)-f(x)\right|^2 dx$.

6) a. Find the complex Fourier series of $f(x)=\sin\frac{x}{2}$ on $[-\pi,\pi]$.

b. Use the above to prove that $\sum_{n=-\infty}^{\infty}\frac{n^2}{(1-4n^2)^2}=\frac{\pi^2}{32}$.

c. Prove the identity $\sum_{n=1}^{\infty}\frac{n^2}{(1-4n^2)^2}=\frac{\pi^2}{64}$.

Answer Key

To view the answers to those exercises, please refer to the appropriate videos on site.

Riemann Lebesgue Lemma

Questions

1) Prove that $\lim_{n\to\infty}\left(n\int_{-\pi}^{\pi}\int_{0}^{x}\frac{se^{s^2}}{\sqrt{s^2+2022}}ds e^{inx}dx\right)=0$.

Answer Key

To view the answer to this exercise, please refer to the appropriate video on site.

Dirichlet's Theorem

Questions

1) Let f be a 2π-periodic function such that $f(x)=\begin{cases} 2+\dfrac{2x}{\pi} & -\pi<x\le 0 \\ 2 & 0\le x<\pi \end{cases}$.

a. Sketch the graph of f on the interval $-3\pi<x<3\pi$.

b. Find the real Fourier series of f.

c. Prove that $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{8}$.

2) Let $f\in L^2_{PC}[-\pi,\pi]$ satisfy $f(x)=x^2$ for $x\in[-\pi,\pi]$.

a. Compute the real Fourier series of f. Use this result to:

b. Compute the sum of the series $\sum_{n=1}^{\infty}\frac{1}{n^4}$.

c. Compute the sum of the series $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$.

d. Compute the sum of the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$.

3) Let $f\in E[-\pi,\pi]$ satisfy $f(x)=\cos(ax)$ for $x\in[-\pi,\pi]$, where $a\notin\mathbb{Z}$.

a. Show that $f(x)\sim\frac{\sin(\pi a)}{\pi a}+\sum_{n=1}^{\infty}\frac{(-1)^n\sin(\pi a)}{\pi}\left(\frac{1}{a+n}+\frac{1}{a-n}\right)\cos(nx)$

b. Use the above to prove the identities:

i. $\csc(\pi a)=\frac{1}{\pi a}+\sum_{n=1}^{\infty}(-1)^n\frac{1}{\pi}\left[\frac{1}{a+n}+\frac{1}{a-n}\right]\quad\left(\csc x=\frac{1}{\sin x}\right)$.

ii. $\cot(\pi a)=\frac{1}{\pi a}+\sum_{n=1}^{\infty}\frac{1}{\pi}\left[\frac{1}{a+n}+\frac{1}{a-n}\right]\quad\left(\cot x=\frac{\cos x}{\sin x}\right)$.

Answer Key

To view the answers to those exercises, please refer to the appropriate videos on site.

Differentiation and Integration of Fourier Series

Questions

1) Prove the following:

<u>Proposition</u>: Let f be piecewise continuously differentiable function on $[-\pi,\pi]$ $\left(\text{i.e. } f' \in L^2_{PC}[-\pi,\pi]\right)$ satisfying $f(\pi)=f(-\pi)$.

Then its complex Fourier series converges absolutely.

2) Let $f(x)=x(\pi-x)$ on the interval $[0,\pi]$.

a. Expand f as a sine series.

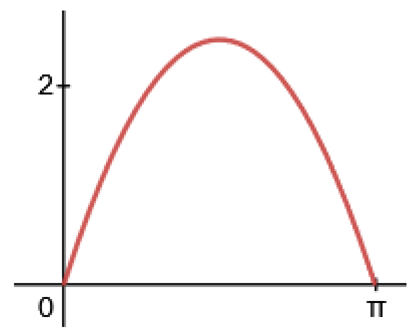
b. To what function does the series converge? Sketch the graph of this function (at least 3 periods).

c. Prove that $\sum_{k=1}^{\infty}\frac{1}{(2k-1)^6}=\frac{\pi^6}{960}$.

d. Prove that $\sum_{n=1}^{\infty}\frac{(-1)^{k+1}}{(2k-1)^3}=\frac{\pi^3}{32}$.

e. Expand $g(x)=\frac{\pi x^2}{2}-\frac{x^3}{3}$ as a cosine series on $[0,\pi]$.

f. Use the above to prove that $\sum_{k=1}^{\infty}\frac{1}{(2k-1)^4}=\frac{\pi^4}{96}$.



3) Let $f(x)=e^{x^2}$ on the interval $[-\pi,\pi]$. Let $f(x)\sim\sum_{-\infty}^{\infty}c_n e^{inx}$ be its complex Fourier expansion.

Prove or disprove:

a. The series $\sum_{-\infty}^{\infty}n^2\left|c_n\right|^2$ converges.

b. The series $\sum_{-\infty}^{\infty}\left|c_n\right|$ converges.

c. The series $\sum_{-\infty}^{\infty}n\left|c_n\right|$ converges.

4) a. Show that the complex series $\sum_{n=0}^{\infty} e^{in(x+i)}$ converges uniformly to some function $f(x)$.

b. How many times can the series $f(x)=\sum_{n=0}^{\infty} e^{in(x+i)}$ be differentiated term–by–term?

5) a. Show that $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n} = \frac{\pi - x}{2}$ for $x \in (0, 2\pi)$. What happens at $x=0,\ x=2\pi$?

b. Let $g(x)=\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^3}$. Compute $g(x)$ explicitly [without series], for $x \in (0, 2\pi)$.

Hint: use term–by–term integration on .a.

6) Let $f:[-\pi,\pi] \to \mathbb{R}$ be $k-1$ times continuously differentiable $(k>0)$ and k times piecewise continuously differentiable.

Suppose that $f^{(j)}(-\pi) = f^{(j)}(\pi)$ for $j=0,1,\ldots,k-1$ and let $f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}$ be the complex Fourier expansion of f. Prove that $\lim_{n\to\infty} n^k c_n = 0$.

7) a. Let $f \in L^2_{PC}[-\pi,\pi]$ be continuously differentiable and suppose f satisfies $f(-\pi)=f(\pi)$ and $\int_{-\pi}^{\pi} f(x)dx = 0$. Prove that $\int_{-\pi}^{\pi} |f(x)|^2 dx \le \int_{-\pi}^{\pi} |f'(x)|^2 dx$.

b. Let $f \in L^2_{PC}[0,\pi]$ be continuously differentiable and suppose f satisfies $f(0)=f(\pi)=0$.

Prove that $\int_{0}^{\pi} |f(x)|^2 dx \le \int_{0}^{\pi} |f'(x)|^2 dx$.

8) Let $f(x)=\sum_{n=1}^{\infty}\frac{1}{1+n^2}e^{in^2x}\quad(-\infty<x<\infty)$.

a. Prove that f is continuous.

b. Prove that f is not continuously differentiable.

9) Let $f:\mathbb{R}\to\mathbb{R}$ be given by $f(x)=\sum_{n=1}^{\infty}\frac{\sqrt{n}}{n^3+1}\sin(n^2x)$.

a. Prove that f is continuous.

b. Prove that f is not continuously differentiable.

10) Let $f:\mathbb{R}\to\mathbb{R}$ be given by $f(x)=\sum_{n=1}^{\infty}\left[\frac{\cos(nx)}{n^{1.4}}+\frac{\sin(nx)}{n^{2.8}}\right]$.

a. Prove that f is continuous.

b. Prove that f is not continuously differentiable.

Answer Key

To view the answers to those exercises, please refer to the appropriate videos on site.

Uniform Convergence of Fourier Series

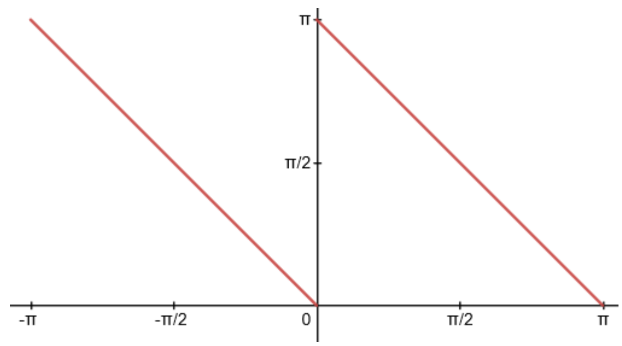
Questions

1) Let $g(x)=\begin{cases}-x & -\pi\le x<0\\ \pi-x & 0\le x<\pi\end{cases}$.

a. Compute the real Fourier series for g.

b. For $-\pi\le x\le\pi$ define $h(x)$ by $h(x)=a\sin\frac{x}{2}+\int_{-\pi}^{x}g(t)dt$ [g is as above].

For which values of a does the Fourier series of h converge uniformly to h on $[-\pi,\pi]$?



2) Let $f(x)=|\sin x|$ in the space $L^2_{PC}[-\pi,\pi]$ and let $f'(x)$ be its derivative.

a. Compute the real Fourier series for f and f'.

b. To which functions do the series (computed above) converge?
Sketch the graphs of these functions on the interval $[-3\pi,3\pi]$.

c. On which closed intervals does the series for f, f' converge uniformly?

Answer Key

To view the answers to those exercises, please refer to the appropriate videos on site.

Fourier Series on a General Interval

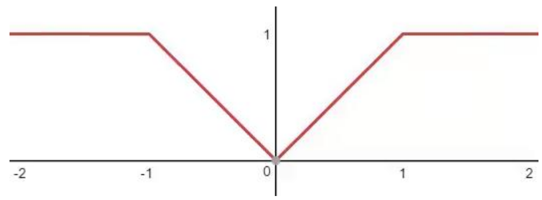
Questions

1) Given the function $f(x)=x^2$ on [0,2π]. Find its real Fourier series.

2) Given the function $f(x)=\min\{1,|x|\}$ on [–2,2].

a. Find the Fourier coefficients a_n, b_n of the real Fourier series of f.

b. Compute $\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}$ and $\sum_{k=0}^{\infty}\frac{1}{(2k+1)^4}$.



3) Given the function $f(x)=e^{\frac{x}{2}}$ on [0,2].

a. Find its complex Fourier series.

b. To what function does the series converge? Sketch its graph (at least 3 periods).

c. Compute the sum of the series $\sum_{n=1}^{\infty}\frac{1}{1+4n^2\pi^2}$.

4) Find the Fourier expansion of the function $f(x)=|x|$ on [–1,1].

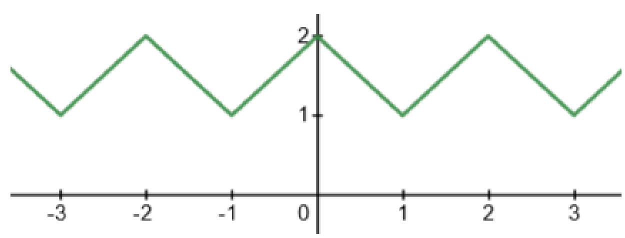
5) Expand the function $f(x)=\begin{cases} x & 0\le x\le 1 \\ 2-x & 1<x<2 \end{cases}$ to a sine–series on [0,2].

6) Let $f:\mathbb{R}\to\mathbb{R}$ be such that $f(x)=f(x+2)$ and $f(x)=2-|x|$ on [–1,1].

a. Expand f as a real Fourier series.

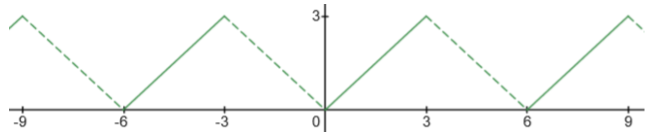
b. Compute the sum $\sum_{k=1}^{\infty}\frac{1}{(2k-1)^4}$.

c. Compute the sum $\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}$.



d. Does the Fourier series for f converge uniformly on [–1,1]?

7) Find the cosine series for $f(x) = x$ on [0,3].



8) Find the sine series for $f(x) = \cos(2x)$ on $[0, \pi]$.

Answer Key

To view the answers to those exercises, please refer to the appropriate videos on site.

Summarizing Exercises

Questions

1) a. Find the complex Fourier series of $f(x)=e^{i\alpha x}$ on $-\pi \le x \le \pi$. Here $\alpha \in \mathbb{R}-\mathbb{Z}$.

b. Show that $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot 2\alpha}{\alpha^2-n^2} = \frac{\pi}{\sin(\pi\alpha)} - \frac{1}{\alpha}$.

c. Show that $\sum_{n=-\infty}^{\infty} \frac{1}{(\alpha-n)^2} = \frac{\pi^2}{\sin^2(\pi\alpha)}$.

d. Show that $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)[(2n+1)^2-\alpha^2]} = \frac{\pi}{4\alpha^2}\left(\sec\frac{\alpha\pi}{2} - 1\right)$.

2) Let $f(x)=|x|$ in the space $E[-\pi,\pi]$ and let $f'(x)$ be it's derivative (piecewise).

a. Find the real Fourier series of f .

b. To which function does the following series converge pointwise on $(-\infty,\infty)$?

$$\sin x + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \frac{1}{7}\sin(7x) + \ldots$$

Compute the sum of the series $\sum_{n=1}^{\infty} \frac{1}{(2n+1)^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots$

3) Let $f \in E[-\pi,\pi]$. Let c_n be its Fourier coefficients (complex) and let $d_n = \mathrm{Re}\{c_n\}$.

Find f , given that it satisfies the following:

a. f is real.

b. f is zero on $[-\pi,0]$.

c. $\sum_{n=-\infty}^{\infty} d_n e^{inx} = x^2 e^{|x|} \cos x$.

Answer Key

To view the answers to those exercises, please refer to the appropriate videos on site.