proprep

Workbook

Table of Contents

Inner Product Spaces, Normed Spaces

Inner Product Spaces, Normed Spaces

Questions

1) $V = C^1[-1,1]$ be the space of continuously differentiable functions on the interval $[-1,1]$.

For any $f, g \in V$ we define $\langle f, g \rangle = \int_{-1}^{1} f(x)\overline{g(x)}dx + \int_{-1}^{1} f'(x)\overline{g'(x)}dx$.

Prove that $\langle .,. \rangle$ is an inner product on V.

2) Let V be an inner product space and let $u, v \in V$ be such that $\langle u, v \rangle = 0$. $u \perp v$

Prove: $\|u+v\|^2 = \|u\|^2 + \|v\|^2$. (Pythagoras)

3) Let $V = C^2[a,b]$ be the space of twice continuously differentiable real functions on $[a,b]$.

I.e. $f \in V$ means that f'' is continuous on $[a,b]$. Define $\langle f, g \rangle = \int_a^b f''(x)g''(x)dx$.

Is this an inner product on V ?

4) Let V be the space of continuously differentiable real functions on $[-1,1]$,

which satisfy the condition $f(-1) = 0$. Define $\langle f, g \rangle = \int_{-1}^{1} f'(x)g'(x)dx$.

Prove that this an inner product on V .

5) Let V be the space of continuous real or complex functions on the interval $[a,b]$.

Prove that the expression $\|f\| = \int_a^b |f(x)|dx$ defines a norm on V .

6) Let V be the space of continuous real or complex functions on the interval $[a,b]$.
Prove that the expression $\|f\| = \max_{[a,b]} |f(x)|$ defines a norm on V.

7) Let V be an inner product space and a normed space (with the induced norm). Let $u, v, w \in V$.
Prove:

a. $\langle u, v+w \rangle = \langle u, v \rangle + \langle u, w \rangle$; $\langle u, \alpha v \rangle = \bar{\alpha} \langle u, v \rangle$

b. $\mathrm{Re}\langle u, v \rangle = \frac{1}{4}\left(\|u+v\|^2 - \|u-v\|^2\right)$

c. $\mathrm{Im}\langle u, v \rangle = \frac{1}{4}\left(\|u+v\|^2 - \|u-v\|^2\right)$

d. $\|u+v\|^2 + \|u-v\|^2 = 2\left\{\|u\|^2 + \|v\|^2\right\}$ (Parallelogram Inequality)

8) Let V be the space of continuous complex functions on the interval $[a,b]$.

Prove that the expression $\|f\| = \int_a^b |f(x)| dx + \max_{[a,b]} |f(x)|$ defines a norm on V.

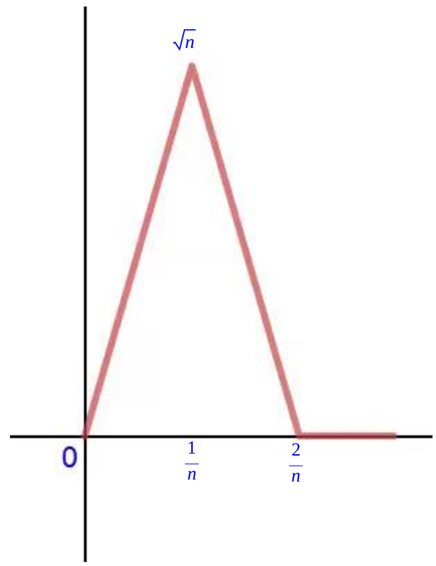
Answer Key

To view the answers to those exercises, please refer to the appropriate videos on site.

Convergence of Functions in Normed Spaces

Questions

1) Let f_n be the following sequence of functions in $C[0,10]$:



$$f_n(x)=\begin{cases} n\sqrt{n}\cdot x & x\in\left[0,\frac{1}{n}\right] \\ 2\sqrt{n}-n\sqrt{n}\cdot x & x\in\left[\frac{1}{n},\frac{2}{n}\right] \\ 0 & x\in\left[\frac{2}{n},10\right] \end{cases}$$

$f_n\left(\frac{1}{n}\right)=\sqrt{n}$

$f_n\left(\frac{2}{n}\right)=0$

a. Does f_n converge pointwise on $[0,10]$?

b. Does f_n converge uniformly on $[0,10]$?

c. Does f_n converge in $L^1[0,10]$?

d. Does f_n converge in $L^2[0,10]$?

2) Given a sequence $f_n(x)=n(1-x)x^n$ of functions in $C[0,1]$.

a. Does f_n converge pointwise on $[0,1]$? If so, find the limit function.

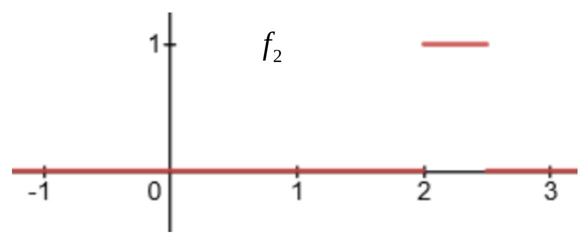
b. Does f_n converge uniformly on $[0,1]$?

c. Does f_n converge in $L^1[0,1]$?

d. Does f_n converge in $L^2[0,1]$?

3) Let $f_n:\mathbb{R}\to\mathbb{R}$ be the defined by

$$f_n(x)=\begin{cases} 1\,;\; x\in\left[n,n+\frac{1}{n}\right] \\ 0\,;\;\text{otherwise} \end{cases}$$



a. Does f_n converge pointwise on $\mathbb{R}$?

b. Does f_n converge uniformly on $\mathbb{R}$?

c. Does f_n converge in the norm of $L^1(-\infty,\infty)$?

d. Does f_n converge in the norm of $L^2(-\infty,\infty)$?

4) Define a sequence $f_n:[1,\infty)\to\mathbb{R}$ by $f_n(x)=[1-\chi_n(x)]\left(x+\frac{1}{n}\right)^{-1}+n^{\alpha}\cdot\chi_n(x)$, where

$\chi_n(x)=\begin{cases}1\ ;\ x\in\left[n-\frac{1}{n^2},n+\frac{1}{n^2}\right]\\ 0\ ;\ \text{otherwise}\end{cases}$ and $\alpha\in\mathbb{R}$ is a parameter.

[χ is the Greek letter chi]

a. For which values of α does f_n converge pointwise on $[1,\infty)$?

b. If the sequence does converge, what is the pointwise limit?

c. For which values of α does f_n converge uniformly on $[1,\infty)$?

d. For which values of α does f_n converge in the norm of $L^1[1,\infty)$?

5) a. Prove that for all $f\in C[a,b]$ we have the inequality $\left|\int_a^x f(t)dt\right|\le\sqrt{x-a}\cdot\|f\|_{L^2[a,b]}$. Reminder:

$\langle f,g\rangle_{L^2[a,b]}=\int_a^b f(t)\overline{g(t)}dt$ and $\|f\|_{L^2[a,b]}=\sqrt{\int_a^b|f(t)|^2\,dt}$.

b. Prove that if $f_n\to f$ in the $L^2[a,b]$ norm then $f_n\to f$ in the $L^1[a,b]$ norm.

Reminder: $\|f\|_{L^1[a,b]}=\int_a^b|f(t)|dt$.

c. Is the converse true? Prove or give a counterexample.

6) a. Let V be a normed space. Prove that for all $u,v\in V$ we have $|u-v|\ge\left|\|u\|-\|v\|\right|$.

b. Prove that if $f_n\xrightarrow{n\to\infty}f$ in norm, then $\|f_n\|\xrightarrow{n\to\infty}\|f\|$.

7) Let V be the vector space of functions $f:[a,b]\to\mathbb{R}$ such that f is differentiable except at a finite number of points and f' is piecewise continuous.

Define $\langle f,g\rangle = f(a)g(a)+\int_a^b f'(x)g'(x)dx$ which is known to be an inner product on V, and let $\|\cdot\|_V$ b the derived norm.

a. If $x_0\in[a,b]$, define the function $g_{x_0}(x)=\begin{cases} x-a+1\ ; & a\le x\le x_0 \\ x_0-a+1; & x_0\le x\le b\end{cases}$,

Prove that $\langle f,g_{x_0}\rangle = f(x_0)$ for all $f\in V$.

b. Prove that $|f(x_0)|\le\sqrt{b-a+1}\cdot\|f\|_V$ for all $f\in V$ and $x_0\in[a,b]$.

c. Let $f_n\in V$ be a sequence of functions which converges in the norm of V to some $f\in V$. Prove that f_n converges to f uniformly.

8) Define a sequence $f_n:\mathbb{R}\to\mathbb{R}$ by $f_n(x)=\sum_{k=1}^{n}\frac{1}{k^2}\chi_{[2^k,2^k+k]}(x)$,

where $\chi_{[a,b]}(x)=\begin{cases}1\ ; & x\in[a,b]\\ 0\ ; & \text{otherwise}\end{cases}$ In general, if $S\subseteq\mathbb{R}$, $\chi_S(x)=\begin{cases}1\ ; & x\in S\\ 0\ ; & \text{otherwise}\end{cases}$

(the characteristic function of *S*).

a. Does the sequence f_n converge in the norm of $L^1(-\infty,\infty)$?

b. Does the sequence f_n converge in the norm of $L^2(-\infty,\infty)$?

9) Let f_n be the following sequence of functions in $L^2[-\pi,\pi]$: $f_n(x)=\sum_{k=1}^{n}\frac{1}{\sqrt{k}}\sin(kx)$.

a. Does the sequence f_n converge in the norm of $L^2[-\pi,\pi]$?

b. Does the sequence f_n converge uniformly on $[-\pi,\pi]$?

Define $h_n(x)=\int_0^x f_n(t)dt$.

c. Does the sequence h_n converge uniformly on $[-\pi,\pi]$?

d. Does the sequence h_n converge in the norm of $L^2[-\pi,\pi]$?

Answer Key

To view the answers to those exercises, please refer to the appropriate videos on site.

Orthonormal Systems

Questions

1) Let V be the space $L^2[-\pi,\pi]$ over $\mathbb{C}$ with inner product $\langle f,g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}dx$.

Let $\varphi_n \in V$ be defined by $\varphi_n(x) = \cos(nx)$.

Prove that the set $\{\varphi_n\}_{n=1}^{\infty}$ is an orthonormal system.

2) Let V be the space $L^2[0,\pi]$ over $\mathbb{C}$ with inner product $\langle f,g\rangle = \frac{2}{\pi}\int_{0}^{\pi} f(x)\overline{g(x)}dx$.

Let $\varphi_n \in V$ be defined by $\varphi_n(x) = \sin(nx)$.

Prove that the set $\{\varphi_n\}_{n=1}^{\infty}$ is an orthonormal system.

3) Let V be an inner product space and let $\{\varphi_n\}_{n=0}^{\infty}$ be a subset of V,

such that $\varphi_n(x)$ is a polynomial of degree n such that $\langle \varphi_n(x), x^m\rangle = 0$ for $m < n$.

Prove that $\{\varphi_n\}_{n=0}^{\infty}$ is an orthogonal set in V.

4) Let V be the space of piecewise continuous functions on the interval $[e^{-\pi}, e^{\pi}]$,

with inner product $\langle f,g\rangle = \int_{e^{-\pi}}^{e^{\pi}} f(x)\overline{g(x)}\frac{1}{x}dx$.

Let $\varphi_n(x) = \sin(n\ln x)$. Prove that $\{\varphi_n\}_{n=1}^{\infty}$ is an orthogonal system in V.

5) Let $\{\varphi_n\}_{n=0}^{\infty}$ be a closed orthonormal set in the space $L^2[a,b]$ with inner product

$\langle f,g\rangle_1 = \int_a^b f(x)\overline{g(x)}dx$. Let $c,d > 0$ be real and define $\psi_n(x) = \sqrt{c}\varphi_n(cx+d)$.

Prove that $\{\psi_n\}_{n=0}^{\infty}$ is a closed orthonormal set in the space $L^2\left[\frac{a-d}{c}, \frac{b-d}{c}\right]$,

with inner product $\langle f,g\rangle_2 = \int_{\frac{a-d}{c}}^{\frac{b-d}{c}} f(x)\overline{g(x)}dx$.

6) Let V be an inner product space over $\mathbb{C}$, let $\{e_1, e_2, \dots, e_N\}$ be a finite orthonormal set in V.

Prove that for all $v \in V$ we have $\left\| \sum_{n=1}^{N} \langle v, e_n \rangle e_n \right\|^2 = \sum_{n=1}^{N} \left| \langle v, e_n \rangle \right|^2$.

7) Let K be the space of piecewise continuous functions $f : (-1,1) \to \mathbb{C}$

with inner product: $\langle f, g \rangle = \int_{-1}^{1} f(x)\overline{g(x)} \underbrace{\frac{1}{\sqrt{1-x^2}}}_{\text{weight func.}} dx$.

Define $T_n(x) = \cos[n \cdot \arccos x]$ $(n = 0, 1, 2, \dots)$.

The T_n are known as Chebyshev [Tchebycheff] polynomials.

Prove that $\{T_n\}_{n=0}^{\infty}$ is an orthogonal system in K and find constants $\alpha_n > 0$,

so that $\{\alpha_n T_n\}_{n=0}^{\infty}$ is an orthonormal system.

$T_0(x) = 1$
$T_1(x) = x$
$T_2(x) = 2x^2 - 1$
$T_3(x) = 4x^3 - 3x$
$T_4(x) = 8x^4 - 8x^2 + 1$
$T_5(x) = 16x^5 - 20x^3 + 5x$
$T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1.$

$\cos(1x)$	$1\cos^1 x$
$\cos(2x)$	$2\cos^2 x - 1$
$\cos(3x)$	$4\cos^3 x - 3\cos^1 x$
$\cos(4x)$	$8\cos^4 x - 8\cos^2 x + 1$
$\cos(5x)$	$16\cos^5 x - 20\cos^3 x + 5\cos^1 x$
$\cos(6x)$	$32\cos^6 x - 48\cos^4 x + 18\cos^2 x - 1$

8) Let K be the space of piecewise continuous functions $f : \mathbb{R} \to \mathbb{C}$, which satisfy the condition $\int_{-\infty}^{\infty} |f(x)|^2 e^{-x^2} dx < \infty$. Inner product: $\langle f, g \rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)} \underbrace{e^{-x^2}}_{\text{weight func.}} dx$. Define

$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left[e^{-x^2}\right]$ (Rodrigues' formula). $\{H_n\}_{n=0}^{\infty}$ are called Hermite polynomials.

Prove that $\{H_n\}_{n=0}^{\infty}$ is an orthogonal system in K and find normalizing constants for them. Useful formula: $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$.

9) Let K be the space of piecewise continuous functions $f:(-1,1)\to\mathbb{C}$,

with inner product: $\langle f,g\rangle=\int_{-1}^{1} f(x)\overline{g(x)}dx$. Define $P_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}\left[(x^2-1)^n\right]$ (Rodrigues' formula).

The P_n are known as Legendre polynomials.

Prove that $\{P_n\}_{n=0}^{\infty}$ is an orthogonal system in K and show that $\langle P_n,P_n\rangle=\frac{2}{2n+1}$ for $n=0,1,2,\ldots$

n	$P_n(x)$
0	1
1	x
2	$\frac{1}{2}(3x^2-1)$
3	$\frac{1}{2}(5x^3-3x)$
4	$\frac{1}{8}(35x^4-30x^2+3)$
5	$\frac{1}{8}(63x^5-70x^3+15x)$
6	$\frac{1}{16}(231x^6-315x^4+105x^2-5)$
7	$1\ (429x^7-693x^5+315x^3-35x)$

10) Let K be the space of piecewise continuous functions $f:\mathbb{R}\to\mathbb{C}$, which satisfy the condition

$\int_0^{\infty}|f(x)|^2 e^{-x}dx<\infty$. The inner product is: $\langle f,g\rangle=\int_0^{\infty} f(x)\overline{g(x)}\underbrace{e^{-x}}_{\text{weight func.}}dx$.

Define $L_n(x)=\frac{1}{n!}e^x\frac{d^n}{dx^n}\left[x^n e^{-x}\right]$ (Rodrigues' formula).

$\{L_n\}_{n=0}^{\infty}$ are called Laguerre polynomials.

Prove that $\{L_n\}_{n=0}^{\infty}$ is an ortho<u>normal</u> system in K. Useful formula: $\int_0^{\infty}x^n e^{-x}dx=n!$.

Answer Key

To view the answers to those exercises, please refer to the appropriate videos on site.

Summarizing Exercises

Questions

1) Let V be the space of piecewise continuous $f:[-\pi,\pi]\to\mathbb{C}$ with inner product,

(no need to prove), given by: $\langle f,g\rangle=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}dx+\frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\overline{g'(x)}dx$.

a. Prove that the set $\left\{e^{inx}\right\}_{n=-\infty}^{n=\infty}$ is an orthogonal system on V and find $\left\|e^{inx}\right\|$, where the norm is the one induced by the inner product.

b. Prove that there does not exist $f\in V$ such that $\lim_{n\to\infty}\frac{n\left|\int_{-\pi}^{\pi}\left[f(x)-in\cdot f'(x)\right]e^{-inx}dx\right|^2}{1+n^2}=1$.

Answer Key

To view the answers to those exercises, please refer to the appropriate videos on site.