proprep

Workbook

Table of Contents

Sequences

Convergence of a Sequence, Monotone Sequences

Questions

1) Let *A* be a non-empty subset of $\mathbb{R}$ and $\alpha = \inf A$. Show that there exists a sequence (a_n) such that an $a_n \in A$ for all $n \in \mathbb{N}$ and $a_n \to \alpha$.

2) Let $x_0 \in \mathbb{Q}$. Show that there exists a sequence (x_n) of irrational numbers such that $x_n \to x_0$.

3) Let (x_n) be a sequence in $\mathbb{R}$. Prove or disprove the following statements.

a. If $x_n \to 0$ and (y_n) is a bounded sequence then $x_n y_n \to 0$.

b. If $x_n \to \infty$ and (y_n) is a bounded sequence then $x_n y_n \to \infty$.

4) Let (x_n) be a sequence in $\mathbb{R}$. Prove or disprove the following statements.

a. If the sequence $(x_n + \frac{1}{n} x_n)$ converges then (x_n) converges.

b. If the sequence $(x_n{}^2 + \frac{1}{n} x_n)$ converges then (x_n) converges.

5) Let $0 < b_1 < a_1$ and define $a_{n+1} = \frac{a_n + b_n}{2}$ and $b_{n+1} = \sqrt{a_n b_n}$ for all $n \in \mathbb{N}$.

Show that both (a_n) and (b_n) converge and that $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n$.

Hint: use the AM-GM inequality: $0 < x < y \Rightarrow \sqrt{xy} < \frac{x+y}{2}$.

6) Let $a > 0$ and $x_1 > 0$. Define (x_n) recursively as $x_{n+1} = \frac{1}{2}(x_n + \frac{a}{x_n})$ for all $n \in \mathbb{N}$.

Show that the sequence (x_n) converges to $\sqrt{a}$.

7) Let (x_n) be a sequence in $(0,1)$. Suppose $4x_n(1 - x_{n+1}) > 1$ for all $n \in \mathbb{N}$.

Show that the sequence is monotone and find the limit.

8) Let *A* be a non-empty subset of $\mathbb{R}$ and $x_0 \in \mathbb{R}$. Show that there exists a sequence (a_n) in *A* such that $|x_0 - a_n| \to d(x_0, A)$. Recall that $d(x, A) = \inf\{|x-a| : a \in A\}$.

9) Let (a_k) be a bounded sequence. For every $n \in \mathbb{N}$, define $x_n = \sup\{a_k : k < n\}$.

Show that the sequence (x_n) converges.

Answer Key

To view the answers to those exercises, please refer to the appropriate videos on site.

Cauchy Criterion, Bolzano - Weierstrass Theorem

Questions

1) Show that the sequence (x_n) defined below satisfies the Cauchy criterion.

a. $x_1 = 1$ and $x_{n+1} = 1 + \frac{1}{x_n}$ for $n \geq 1$.

b. $x_1 = 1$ and $x_{n+1} = \frac{1}{2+x_n^2}$ for $n \geq 1$.

c. $x_1 = 1$ and $x_{n+1} = \frac{1}{6}(x_n^2 + 8)$ for $n \geq 1$.

2) Let (x_n) be a sequence of positive real numbers. Prove or disprove the following statements.

a. If $x_{n+1} - x_n \to 0$ then (x_n) converges.

b. If $|x_{n+2} - x_{n+1}| < |x_{n+1} - x_n|$ for all $n \in \mathbb{N}$ then (x_n) converges.

c. If (x_n) satisfies the Cauchy criterion, then there exists an $\alpha \in \mathbb{R}$ such that $0 < \alpha < 1$ and $|x_{n+2} - x_{n+1}| \leq \alpha \cdot |x_{n+1} - x_n|$ for all $n \in \mathbb{N}$.

3) Let (x_n) be a sequence of integers such that $|x_{n+1} - x_n| \geq 1$ for all $n \in \mathbb{N}$.
Prove or disprove the following statements.

a. The sequence (x_n) does not satisfy the Cauchy criterion.

b. The sequence (x_n) cannot have a convergent subsequence.

4) Suppose that $0 < \alpha < 1$ and that (x_n) is a sequence satisfying the condition:

$|x_{n+1} - x_n| \leq \alpha^n$, $n = 1, 2, 3...$ Show that (x_n) satisfies the Cauchy criterion.

5) Let $1 \leq x_1 \leq x_2 \leq 2$ and $x_{n+2} = \sqrt{x_{n+1}x_n}$, $n \in \mathbb{N}$. Show that:

a. $\frac{x_{n+1}}{x_n} \geq \frac{1}{2}$ for all $n \in \mathbb{N}$.

b. $|x_{n+2} - x_{n+1}| \leq \frac{2}{3}|x_{n+1} - x_n|$.

c. (x_n) converges.

6) Show that a sequence (x_n) of real numbers has no convergent subsequence if and only if $|x_n| \to \infty$.

7) Let (x_n) be a sequence in $\mathbb{R}$ and $x_0 \in \mathbb{R}$. Suppose that every subsequence of (x_n) has a subsequence converging to x_0. Show that $x_n \to x_0$.

8) Let (x_n) be a sequence in $\mathbb{R}$. We say that a positive integer *n* is a peak of the sequence if $m > n$ implies $x_n > x_m$ (i.e., if x_n is greater than every subsequent term in the sequence).

a. If (x_n) has infinitely many peaks, show that it has a decreasing subsequence.

b. If (x_n) has only finitely many peaks, show that it has an increasing subsequence.

c. From (a) and (b) conclude that every sequence in $\mathbb{R}$ has a monotone subsequence. Further, every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

(An alternate proof of Bolzano-Weierstrass Theorem).

9) Let (x_n) be a sequence defined by $x_1 = 1,\ x_2 = 2 \text{ and } x_{n+2} = \frac{3}{4}x_n + \frac{1}{4}x_{n+1} \text{ for } n \geq 1$.

a. Show that (x_n) converges.

b. Find the limit of (x_n).

10) Show that there is no continuous function which maps $[0,1]$ onto $(0,1)$.

11) Let $p(x) = a + bx + cx^2$. Find all values of $a, b, c \in \mathbb{R}$ for which the function $p(|x|)$ is differentiable at 0.

Answer Key

To view the answers to those exercises, please refer to the appropriate videos on site.

Sequence with No Limit

Questions

1) Prove that $\lim_{n\to\infty}\frac{4n+1}{n+2}\neq 3$ using the definition of a limit.

2) Prove that $\lim_{n\to\infty}\frac{n+10}{4n+2}\neq \frac{1}{2}$ using the definition of a limit.

3) Prove that $\lim_{n\to\infty}\frac{4n^2+n+1}{2n^2+2}\neq 1$ using the definition of a limit.

4) Prove that $\lim_{n\to\infty}\frac{4n^2+n+1}{2n^2+4n+2}\neq 1$ using the definition of a limit.

5) Prove that $\lim_{n\to\infty}\left[(-1)^n\frac{4n+1}{n+2}\right]\neq 4$ using the definition of a limit.

Answer Key

To view the answers to those exercises, please refer to the appropriate videos on site.