proprep

Workbook

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The Limit of a Function

Technique 1 – Substitution

Questions

Find the following limits if possible:

1) $\lim_{x\to 4} x^2 + x + 1$ **2)** $\lim_{x\to 10} \frac{x+1}{x+2}$ **3)** $\lim_{x\to 1^+} \sqrt{x+3}$ **4)** $\lim_{x\to 100} 20$

Answer Key

1) 21 **2)** $\frac{11}{12}$ **3)** 2 **4)** 20

Technique 2 – Factoring

Questions

Find the following limits if possible:

1) $\lim_{x\to 3}\dfrac{x^2-x-6}{x^2-9}$

2) $\lim_{x\to -5}\dfrac{2x^2-50}{2x^2+3x-35}$

3) $\lim_{x\to 1}\dfrac{x^7-x}{x-1}$

4) $\lim_{x\to 1}\dfrac{x^n-x}{x-1}\quad n>1$

Answer Key

1) $\dfrac{5}{6}$ **2)** $\dfrac{20}{17}$ **3)** 6 **4)** $n-1$

Technique 3 – Multiplication by Conjugate

Questions

Calculate the following limits if possible:

1) $\lim_{x\to 1}\dfrac{1-\sqrt{x}}{1-x}$

2) $\lim_{x\to 3}\dfrac{x-3}{\sqrt{x+1}-2}$

3) $\lim_{x\to 3}\dfrac{3-\sqrt{x+6}}{2x-6}$

4) $\lim_{x\to 1}\dfrac{\sqrt{x^2+x+2}-2}{x^2-1}$

5) $\lim_{x\to 4}\dfrac{\sqrt{2x+1}-\sqrt{x+5}}{x-4}$

6) $\lim_{x\to 1}\dfrac{2-\sqrt{3x+1}}{1-\sqrt{x-1}}$

7) $\lim_{x\to 1}\dfrac{1-\sqrt[3]{x}}{1-x}$

Answer Key

1) $\dfrac{1}{2}$ **2)** 4 **3)** $-\dfrac{1}{12}$ **4)** $\dfrac{3}{8}$

5) $\dfrac{1}{6}$ **6)** $\dfrac{3}{4}$ **7)** $\dfrac{1}{3}$

Technique 4 – $f(x)$ Tends to Infinity

Questions

Calculate the following limits if possible:

1) $\lim_{x\to 0}\dfrac{x^2+4}{x}$

2) $\lim_{x\to 2}\dfrac{(x-1)^2}{x-2}$

3) $\lim_{x\to 2}\dfrac{x^2-1}{(x-2)(x-5)}$

4) $\lim_{x\to 0^+}\dfrac{\ln x}{x}$

5) $\lim_{x\to 2^-}-\dfrac{1}{2}\ln(2-x)$

6) $\lim_{x\to 0^+}\left((\ln x)^2+x(\ln x-3)\right)$

7) $\lim_{x\to 0}e^{\frac{1}{x}}$

8) $\lim_{x\to 0^+}\dfrac{1}{1+2^{\frac{1}{x}}}$

9) $\lim_{x\to 0^-}\dfrac{1}{1+2^{\frac{1}{x}}}$

10) $\lim_{x\to 0}\dfrac{1}{1+2^{\frac{1}{x}}}$

11) $\lim_{x\to 0^+}\ln x\cdot\cot x$

Answer Key

1) No limit **2)** No limit **3)** No limit **4)** $-\infty$ **5)** ∞ **6)** ∞

7) No limit **8)** 0 **9)** 1 **10)** No limit **11)** $-\infty$

Technique 5 – x Tends to Infinity

Questions

Find the following limits if possible:

1) $\lim_{x\to\infty}\left(e^{-x}\right)^{\ln x}$

2) $\lim_{x\to-\infty}\arctan\left(x\right)+e^{x}$

3) $\lim_{x\to\infty}\frac{4x^2+2}{x^2+1000x}$

4) $\lim_{x\to-\infty}\frac{x^4+2x^2+6}{3x^2+10x}$

5) $\lim_{x\to\infty}\frac{x^4+2x^2+6}{3x^5+10x}$

6) $\lim_{x\to\infty}\left(\frac{x^2-5x+6}{2x+10}-\frac{x}{2}\right)$

7) $\lim_{x\to\infty}\frac{\sqrt{x^2+1}}{x}$

8) $\lim_{x\to-\infty}\frac{\sqrt{x^2+1}}{x}$

9) $\lim_{x\to-\infty}\frac{\sqrt{9x^6-5x}}{x^3-2x^2+1}$

10) $\lim_{x\to\infty}\frac{\sqrt[3]{x^4+2x^2+6+27x^6}}{\sqrt{3x^3+10x+4x^4}}$

11) $\lim_{x\to\infty}\frac{\sqrt{x+2}-\sqrt{3x-3}}{\sqrt{4x+1}-\sqrt{5x-1}}$

12) $\lim_{x\to-\infty}\frac{16^{x}+4^{x+1}}{2^{4x+2}+2^{x+3}}$

13) $\lim_{x\to\infty}\frac{4\cdot9^{x}+3^{x+1}}{81^{0.5x}+3^{x+3}}$

14) $\lim_{x\to-\infty}\frac{4\cdot9^{x}+3^{x+1}}{81^{0.5x}+3^{x+3}}$

15) $\lim_{x\to\infty}\sqrt{\frac{4x^2+2}{x^2+1000x}}$

16) $\lim_{x\to\infty}\ln\left(\frac{3x^3-5x-1}{x^3-2x^2+1}\right)$

17) $\lim_{x\to\infty}e^{\frac{x^4+2x^2+6}{3x^4+10x}}$

18) $\lim_{x\to-\infty}\sin\left(\frac{x^4+2x^2+6}{3x^5+10x}\right)$

19) $\lim_{x\to\infty}\left(\sqrt{x^2+kx}-x\right)$

20) $\lim_{x\to\infty}\left(\sqrt{x^2+x+1}-x\right)$

21) $\lim_{x\to-\infty}\left(\sqrt{x^2+x+1}+x\right)$

22) $\lim_{x\to\infty}\left(\sqrt{x^4+x^2+1}-x^2\right)$

23) $\lim_{x\to\infty}\left(\sqrt{x^2+ax}-\sqrt{x^2+bx}\right)$

Answer Key

1) 0

2) $-\frac{\pi}{2}$

3) 4

4) $-\infty$

5) 0

6) -5

7) 1

8) -1

9) -3

10) $\frac{3}{2}$

11) $\frac{1-\sqrt{3}}{2-\sqrt{3}}$

12) 0

13) 4

14) $\frac{1}{9}$

15) 2

16) $\ln 3$

17) $e^{\frac{1}{3}}$

18) 0

19) $\frac{k}{2}$

20) $\frac{1}{2}$

21) $-\frac{1}{2}$

22) $\frac{1}{2}$

23) $\frac{a-b}{2}$

Technique 6 – Euler's Limit

Questions

Find the following limits if possible:

1) $\lim_{x\to-\infty}\left(1+\frac{1}{2x}\right)^{x}$

2) $\lim_{x\to-\infty}\left(1+\frac{1}{x^2}\right)^{x}$

3) $\lim_{x\to\infty}\left(\frac{x+2}{x}\right)^{x}$

4) $\lim_{x\to\infty}\left(1-\frac{1}{x^2}\right)^{x^2-1}$

5) $\lim_{x\to\infty}\left(\frac{2x+3}{2x-3}\right)^{x}$

6) $\lim_{x\to0}(1+\sin x)^{\frac{1}{x}}$

7) $\lim_{x\to\infty}\left(\frac{2x+3}{2x-3}\right)^{x}$

8) $\lim_{x\to\infty}\left(\frac{x^2+x+1}{x^2+x+4}\right)^{4x^2}$

9) $\lim_{x\to\infty}\left(1+\tan\frac{1}{x}\right)^{x}$

Answer Key

1) $\sqrt{e}$ **2)** 1 **3)** e^2 **4)** e^{-1} **5)** e^3

6) e **7)** e^3 **8)** e^{-12} **9)** e

Technique 7 – Trigonometric Limits

Questions

Find the following limits if possible:

1) $\lim_{x\to 0}\frac{\sin(3x)}{4x}$

2) $\lim_{x\to 0}\frac{\sin(3x)}{\sin(4x)}$

3) $\lim_{x\to 0}\frac{x\cos(x)}{\sin(2x)}$

4) $\lim_{x\to 0}\frac{1-\cos(x)}{x^2}$

5) $\lim_{x\to 0}\frac{\tan(x)\sin(x)}{x}$

6) $\lim_{x\to 0}\frac{\sqrt{1-\sin(x)}-\sqrt{\cos(x)}}{x}$

7) $\lim_{x\to 0}\frac{1-\cos(1-\cos(x))}{x^4}$

8) $\lim_{x\to 0}\frac{3\sin(x)-\sin(3x)}{x^3}$

9) $\lim_{x\to 0}\frac{1-\sqrt{\cos(x)}}{x^2}$

Answer Key

1) $\frac{3}{4}$ **2)** $\frac{3}{4}$ **3)** $\frac{1}{2}$ **4)** $\frac{1}{2}$ **5)** $\frac{1}{2}$

6) $\frac{1}{2}$ **7)** $\frac{1}{8}$ **8)** 4 **9)** $\frac{1}{4}$

Technique 8 – The Sandwich/Squeeze Theorem

Questions

Find the following limits if possible:

1) $\lim_{x\to\infty}\frac{\sin x}{x}$

2) $\lim_{x\to\infty}\frac{\cos(2x+1)}{x}$

3) $\lim_{x\to\infty}\frac{3x+\sin x}{4x+\cos x}$

4) $\lim_{x\to\infty}\frac{3x^2+x+\sin 2x}{x^2+\cos 3x}$

5) $\lim_{x\to 0} x\cdot\sin\left(\frac{1}{x}\right)$

6) $\lim_{x\to 0} x^2\cdot\cos\left(\ln x^2\right)$

7) $\lim_{x\to\infty}\frac{3x+\arctan(2x-3)}{4x+\arctan(x-\ln x)}$

8) $\lim_{x\to\infty}\sqrt[x]{2^x+3^x+4^x}$

9) $\lim_{x\to\infty}\frac{1}{x}[x]$

Answer Key

1) 0 **2)** 0 **3)** $\frac{3}{4}$

4) 3 **5)** 0 **6)** 0

7) $\frac{3}{4}$ **8)** 4 **9)** 1

Technique 9 – Piecewise Functions

Questions

Find the following limits if possible:

1) $\lim_{x\to 0} f(x)\ ;\ f(x)=\begin{cases}\dfrac{\sin 4x}{x} & x>0\\ 4+e^{\frac{1}{x}} & x<0\end{cases}$

2) $\lim_{x\to 1} f(x)\ ;\ f(x)=\begin{cases}\dfrac{x^2+x-2}{x-1} & x>1\\ \dfrac{x-1}{\sqrt{x}-1} & x<1\end{cases}$

3) $\lim_{x\to 0}\dfrac{|x|}{x}$

4) $\lim_{x\to \infty}\dfrac{|x|}{x}$

5) $\lim_{x\to -\infty}\dfrac{|x|}{x}$

Answer Key

1) 4
2) No limit
3) No limit
4) 1
5) -1

Limit from Definition

Questions

1) Use the definition of the limit to prove that $\lim_{x\to 2}(7x+14)=28$.

2) Use the definition of the limit to prove that $\lim_{x\to 3}x^2=9$.

3) Use the definition of the limit to prove that $\lim_{x\to 1}\left(x^2-1\right)=0$.

4) Use the definition of the limit to prove that $\lim_{x\to 24}\sqrt{x+1}=5$.

5) Use the definition of the limit to prove that $\lim_{x\to 1}\frac{1}{x}=1$.

6) Use the definition of the limit to prove that $\lim_{x\to \frac{\pi}{4}}\sin x=\sin\frac{\pi}{4}$.

7) Use the definition of the limit to prove that $\lim_{x\to 2}\frac{3+x}{x^2+1}=1$.

8) Use the definition of the limit to prove that $\lim_{x\to 4^-}\left(\sqrt{4-x}\right)=0$.

9) Use the definition of the limit to prove that:

a. $\lim_{x\to 0^+}\frac{|x|}{x}=1$

b. $\lim_{x\to 0^-}\frac{|x|}{x}=-1$

10) Use the definition of the limit to prove that $\lim_{x\to 2}\frac{-5}{(x-2)^2}=-\infty$.

11) Use the definition of the limit to prove that $\lim_{x\to 3^-}\frac{-2}{x-3}=\infty$.

12) Use the definition of the limit to prove that $\lim_{x\to 0^+}\ln x=-\infty$.

13) Use the definition of the limit to prove that $\lim_{x\to\infty}\frac{x+7}{x+2}=1$.

14) Use the definition of the limit to prove that $\lim_{x\to\infty}\frac{3-4x}{2x+1}=-2$.

15) Use the definition of the limit to prove that $\lim_{x\to\infty}\frac{3x^2-1}{x^2+x+1}=3$.

16) Given a function $f(x)$ which satisfies $\lim_{x\to\infty} f(x)=-5$.

Prove that there exists an $M>0$ such that $f(x)<-4$ whenever $x>M$.

17) Given a function $f(x)$ which satisfies $\lim_{x\to-\infty} f(x)=5$.

Prove that there exists an $M<0$ such that $f(x)>4$ whenever $x<M$.

18) Given a positive function $f(x)$ on the interval $[a,\infty)$ which satisfies $\lim_{x\to\infty} f(x)=0$.

Prove [using *epsilon-delta*] that $\lim_{x\to\infty}\sqrt{f(x)}=0$.

19) Given the limit: $\lim_{x\to\infty}\underbrace{\frac{x^2+2x}{x^2+3x+2}}_{f(x)}=\underbrace{1}_{L}$ *(no need to prove)*.

Find a value for $M>0$ such that $|f(x)-L|<\underbrace{0.1}_{\varepsilon}$ whenever $x>M$.

20) Use the definition of continuity to prove that $f(x)=2x-3$ is continuous at $x=4$.

Answer Key:

1-18) and **20)** To see the answers ro those exercises, please refer to the appropriate videos.
19) $x>30$