proprep

Workbook

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Analytic Functions

Complex Functions

Questions

1) Express the complex function $f(z)=z\cdot\operatorname{Re}(z)$ in the form $f(\underbrace{x+iy}_{z})=u(x,y)+i\cdot v(x,y)$.

2) Express the complex function $f(z)=|z|^2$ in the form $f(\underbrace{x+iy}_{z})=u(x,y)+i\cdot v(x,y)$.

3) Express the complex function $f(z)=2|z|^2+i(\bar{z})^2$ in the form $f(\underbrace{x+iy}_{z})=u(x,y)+i\cdot v(x,y)$.

4) Express the complex function $f(z)=\dfrac{z}{1+|z|^2}$ in the form $f(\underbrace{x+iy}_{z})=u(x,y)+i\cdot v(x,y)$.

5) Express the complex function $f(z)=z^2+\bar{z}$ in the form $f(\underbrace{x+iy}_{z})=u(x,y)+i\cdot v(x,y)$.

6) Express the complex function $f(\underbrace{x+iy}_{z})=\dfrac{x^3}{3}-i\cdot\dfrac{y^3}{3}$ in the form $f(z)=expression\ in\ z$.

Answer Key

1) $f\left(x+iy\right)=\underbrace{x^2}_{u}+i\cdot\underbrace{xy}_{v}$

2) $f\left(x+iy\right)=\underbrace{x^2+y^2}_{u}+i\cdot\underbrace{0}_{v}$

3) $f\left(x+iy\right)=\underbrace{2\left(x^2+xy+y^2\right)}_{u}+i\cdot\underbrace{\left(x^2-y^2\right)}_{v}$

4) $f\left(x+iy\right)=\underbrace{\frac{x}{1+x^2+y^2}}_{u}+i\cdot\underbrace{\frac{y}{1+x^2+y^2}}_{v}$

5) $f\left(x+iy\right)=\underbrace{x^2+x-y^2}_{u}+i\cdot\underbrace{(2xy-y)}_{v}$

6) $f\left(z\right)=\frac{1}{12}\left(z^3+3z\cdot\overline{z}^2\right)$ or $\frac{z^3+3z\cdot\overline{z}^2}{12}$ or $\frac{1}{12}z^3+\frac{1}{4}z\cdot\overline{z}^2$

Complex Limits and Continuity

Questions

1) Evaluate the limit $\lim_{z\to 0}\frac{\overline{z}}{z}=?$ if it exists.

2) Evaluate the limit $\lim_{z\to 0}\frac{z^4}{|z|^4}=?$ if it exists.

Answer Key

1) The limit does not exist.

2) The limit does not exist.

Complex Limits and Continuity

Questions

1) Given the function $f(z)=\overline{z},\ z\in\mathbb{C}$.
At which points, if any, is *f* differentiable?

2) Given the function $f(z)=\mathrm{Re}(z),\ z\in\mathbb{C}$.
At which points, if any, is *f* differentiable?

3) Given the function $f(z)=|z|^2,\ z\in\mathbb{C}$.
At which points, if any, is *f* differentiable?

Answer Key

1) $\varnothing$ f is nowhere differentiable.
2) $\varnothing$ f is nowhere differentiable.
3) Z = 0

Cauchy Riemann Equations

Questions

1) Show that the function $f(z)=z^2+\text{Im}(z)$ is not differentiable anywhere.

2) Show that the function $f(z)=f(x+iy)=xy+i(x^2+y^2)$ is differentiable only at $z=0$.

3) Find real values for the parameters a, b such that the function.
$f(z)=e^{ax}\cos(3y)-ie^{-3x}\sin(by)$ is differentiable everywhere.

4) It is known that the function $f(z)=\frac{z}{\bar{z}}$ is not continuous at $z=0$.
Find all points *z* (if any) where *f* is differentiable.

5) Let $f(z)$ be differentiable in a domain *D* and suppose that $\text{Re}\{f(z)\}=0 \ \ \forall z \in D$.
Prove that $f(z)$ is constant.

6) Let $f(z)$ be differentiable and not constant in a domain *D*.
Define $g(z)=\overline{f(z)} \ \ \forall z \in D$.
Prove that $g(z)$ can't be differentiable everywhere on *D*.

7) Let $f(z)$ be piecewise defined as follows: $f(z)=\begin{cases} e^{-\frac{1}{z^4}} & ; z \neq 0 \\ 0 & ; z=0 \end{cases}$

a. Prove that *f* is discontinuous at $z=0$
b. Show that the C-R equations hold at $z=0$

8) Let $f(z)$ be analytic on the domain $H^+=\{z \in \mathbb{C} \,|\, \text{Im}(z)>0\}$ [upper half-plane].
Prove: $g(z)=\overline{f(\bar{z})}$ is analytic on $H^-=\{z \in \mathbb{C} \,|\, \text{Im}(z)<0\}$ [lower half-plane].

9) Show that $f(z)=e^{\text{Re}(z)}$ is not differentiable anywhere.

10) Given the function $f(z)=cx^2-xy+ixy^2$ where *c* is a complex constant.

a. Given that *f* is differentiable at $z=i+1$, find the value of *c*.
b. Using this *c*, find all points at which *f* is differentiable.

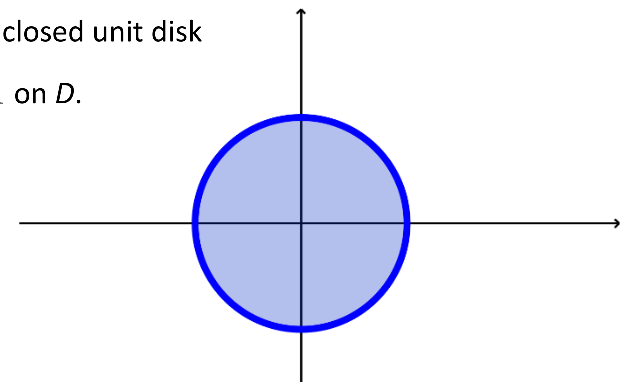
11) The function $f(z)=\frac{1}{2}\ln\left(x^2+y^2\right)+i\cdot\arctan\left(\frac{y}{x}\right)$ is defined on the right half-plane $H=\left\{z\in\mathbb{C}\mid \overbrace{\mathrm{Re}(z)}^{x}>0\right\}$. Is *f* analytic on *H*?

12) Given the function $f(z)=e^{\frac{x^2-y^2}{2}}\left[\cos(xy)+i\cdot a\sin(xy)\right]$ where *a* is a real parameter. For which values of *a* is *f* holomorphic in the whole plane?

13) Suppose the function $g(z)$ is holomorphic on the closed unit disk $D=\overline{D(0,1)}=\left\{z\in\mathbb{C}\mid |z|\le 1\right\}$ and satisfies $|g(z)|=1$ on *D*.

Prove that *g* is constant on *D*.

Hint: express *g* as $g(z)=e^{i\cdot h(x,y)}$

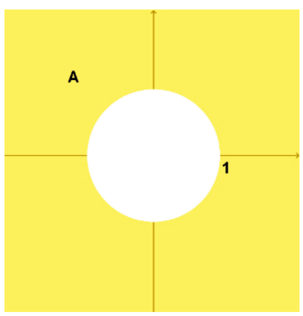


14) Let $D=D(0,R)=\left\{z\in\mathbb{C}\mid |z|<R,\ R>0\right\}$ and let $f:D\to\mathbb{C}$ be analytic on all of *D*.

Let $g(z)=\overline{f\left(\frac{R}{\bar{z}}\right)}$. Find a suitable domain *A* for *g*, and check if *g* is differentiable on *A*.

Answer Key

1) Proof
2) Proof
3) $a=-3,\ b=3$
4) $\varnothing$ f is differentiable nowhere!
5) Proof
6) Proof
7) Proof
8) Proof
9) Proof
10) a. $c=\frac{3}{2}$ b. $0,\quad 1+i,\quad 0.25+0.5i$
11) Yes
12) $a=1$
13) Proof
14) $A=\{z\,|\,|z|>1\}$



Harmonic Functions

Questions

1) Show that the function $u(x,y)=x^3-3xy^2$ is harmonic in the whole plane.

2) Show that the function $u(x,y)=x^2-y^2$ is harmonic in the whole plane, and find a harmonic conjugate *v* of *u*.

3) Show that the function $f(z)=xy+i(x^2+y^2)$ is differentiable at the origin *O* but that its imaginary part is not harmonic at *O*. Is $f(z)$ holomorphic at *O?*

4) Show that the function $u(x,y)=\sin(x)\cosh(y)$ is harmonic in the entire plane and find its conjugate of $v(x,y)$ which satisfies $v(0,0)=2$.
Hint: consider the function $f(z)=\sin z$,
and use the formula $\sin(x+iy)=\sin x\cosh y+i\cos x\sinh y$

5) Show that the function $u(x,y)=\cos(x)\sinh(y)$ is harmonic in the entire plane, and find a holomorphic [analytic] $f(z)$ such that $u(x,y)=\mathrm{Re}\{f(x+iy)\}$.
Hint: refer to the previous exercise.

6) Show that the function $v(x,y)=e^y\sin x$ is harmonic in the entire plane.
Find a conjugate *u* of $-v$ and an entire function *f* such that $f(x+iy)=u(x,y)+iv(x,y)$.

7) Show that the polar function $u(r,\theta)=\left(r+\frac{1}{r}\right)\cos\theta$ is harmonic in for $r\neq 0$.

Reminder: In polar form, $u(r,\theta)$ is harmonic iff $r^2u''_{rr}+ru'_r+u''_{\theta\theta}=0$.

8) Given that the polar function $u(r,\theta)=\left(r+\frac{1}{r}\right)\cos\theta$ is harmonic in the domain $r\neq 0$,

find a conjugate $v(r,\theta)$ of $u(r,\theta)$ in this domain.

9) Show that the function $u(x,y)=2x-x^3+3xy^2$ is harmonic, and find a harmonic conjugate $v(x,y)$ of $u(x,y)$.

10) Let $f(\underbrace{x+iy}_{z})=u(x,y)+iv(x,y)$ be an entire function.

Prove that $g(x,y)=u(x,y)^2-v(x,y)^2$ is harmonic everywhere.

11) Let $f(\underbrace{x+iy}_{z})=u(x,y)+iv(x,y)$ be an entire function.

Prove that $g(x,y)=\sin u(x,y)\cdot\cosh v(x,y)$ is harmonic everywhere.

12) Find (if any) all harmonic functions of the form $u(x,y)=\varphi\left(\frac{x^2+y^2}{x}\right)$ $(x\neq 0)$,

where $\varphi=\varphi(t)$ is a real-valued function with a continuous 2nd order derivative.

13) Find (if any) all harmonic functions of the form $u(x,y)=\varphi\left(\frac{x}{y}\right)$ $(y\neq 0)$,

where $\varphi=\varphi(t)$ is a real-valued function with a continuous 2nd order derivative.

Answer Key

1) Proof
2) Need to show that that Laplacian $\Delta u = u''_{xx} + u''_{yy} = 0$ in the whole plane: $v(x,y) = 2xy$
3) Proof
4) $v(x,y) = \cos x \sinh y + 2$
5) $f(z) = -i \sin z$
6) $u(x,y) = -e^{y} \cos(x)$, $f(z) = -e^{-iz}$
7) Proof
8) $v(r,\theta) = \left(r - \frac{1}{r}\right) \sin\theta$
9) $v(x,y) = 2y - 3x^2 y + y^3$
10) Proof
11) Proof
12) $u(x,y) = -\frac{c_1 \cdot x}{x^2 + y^2} + c_2 \quad (x \neq 0)$
13) $u(x,y) = c_1 \arctan\frac{x}{y} + c_2 \quad (y \neq 0)$