proprep

Workbook

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Complex Series

Numerical Series

Questions

1) Check the convergence of the series $\sum_{n=0}^{\infty} \frac{\cos(in)}{2^n}$.

2) Check the convergence of the series $\sum_{n=0}^{\infty} \frac{n\sin(in)}{3^n}$.

3) Check the convergence of the series $\sum_{n=0}^{\infty} \left(\frac{2+i}{\sqrt{5}}\right)^n$.

4) Check the convergence of the series $\sum_{n=0}^{\infty} \frac{1}{2+i^n}$.

5) Check the convergence of the series $\sum_{n=2}^{\infty} \frac{1}{n^2+i^n}$.

Answer Key

1) Diverges
2) Converges
3) Diverges
4) Diverges
5) Converges

The Cauchy-Hadamard Criterion

Questions

1) Find the radius of convergence of the series $\sum_{n=0}^{\infty} e^{in} z^n$.

2) Find the radius of convergence of the series $\sum_{n=0}^{\infty} \left(\frac{z}{in} \right)^n$.

3) Find the radius of convergence of the series $\sum_{n=0}^{\infty} \frac{1}{3^n (2n+1)} (z-3)^n$.

Answer Key

1) $R = 1$

2) $R = \infty$

3) $R = 3$

General Series

Questions

1) Find the domain of convergence of the series $\sum_{n=0}^{\infty} z^{-n}$.

2) Find the domain of convergence of the series $\sum_{n=0}^{\infty} \frac{z^{-n}}{(1-i)^n}$.

3) Find the domain of convergence of the series $\sum_{n=0}^{\infty} \frac{1}{4^n (z-1)^n}$.

4) Find the domain of convergence of $\sum_{n=0}^{\infty} \left(\frac{2}{z}\right)^n + \sum_{n=0}^{\infty} \left(\frac{z}{4}\right)^n$.

5) Find the domain of convergence of the series $\sum_{n=0}^{\infty} e^{nz}$.

Answer Key

1) $|z|>1$

2) $|z|>\frac{1}{\sqrt{2}}$

3) $|z-1|>\frac{1}{4}$

4) $2<|z|<4$

5) $\operatorname{Re} z<0$

The Weierstrass Test for Uniform Convergence

Questions

1) Let $0 < r < 1$ and let $U = \{z \mid |z| \le r\}$. Prove that the series $\sum_{n=0}^{\infty} z^n$ converges uniformly on *U*.

2) Prove that the series $\sum_{n=0}^{\infty} \frac{z^n}{n^2}$ converges uniformly on $U = \{z \mid |z| \le 1\}$.

3) Prove that the series $\sum_{n=0}^{\infty} \frac{1}{(z^2-1)^n}$ converges uniformly on $U = \{z \mid |z| \ge 2\}$.

4) Let $0 < r < 1$ and let $U = \{z \mid |z| \le r\}$. Prove that the series $\sum_{n=0}^{\infty} \frac{z^{2n}}{z^n+1}$ converges uniformly on *U*.

5) Prove that the series $\sum_{n=0}^{\infty} n^{-z}$ converges uniformly on $U = \{z \mid \operatorname{Re} z \ge 2\}$.

*Proof questions- for the full solution sww the video

Taylor Series

Questions

1) Find the Taylor series for $f(z)=\sin(z+1)$ around $z=0$.
What is its radius of convergence?

2) Find the Taylor series for $f(z)=\frac{1}{z}$ around $z=i$.
What is its radius of convergence?

3) Find the Taylor series for $f(z)=\frac{2i}{2+i+z}$ around $z=z_0$ where $z_0\neq -2-i$ is arbitrary.
What is the domain of convergence of the series?

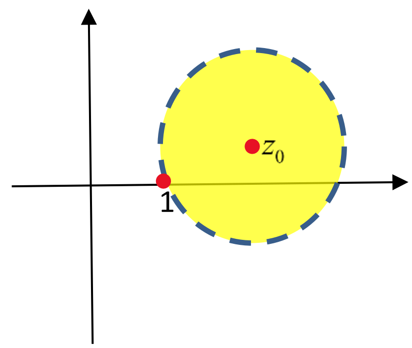
4) Find Taylor series for $f(z)=\frac{1}{(1-z)^3}$ around $z=z_0$ where $z_0\neq 1$ is arbitrary.
What is the domain of convergence of the series?

Answer Key

1) $f(z)=\sum_{n=0}^{\infty}(-1)^n\left(\frac{\sin 1}{(2n)!}z^{2n}+\frac{\cos 1}{(2n+1)!}z^{2n+1}\right)$ Converges on all $\mathbb{C}$.

2) $f(z)=\frac{1}{i}\cdot\sum_{n=0}^{\infty}i^n(z-i)^n$ Converges on $|z-i|<1$.

3) $f(z)=\frac{1}{i}\cdot\sum_{n=0}^{\infty}i^n(z-i)^n$ Converges on $|z-i|<1$.

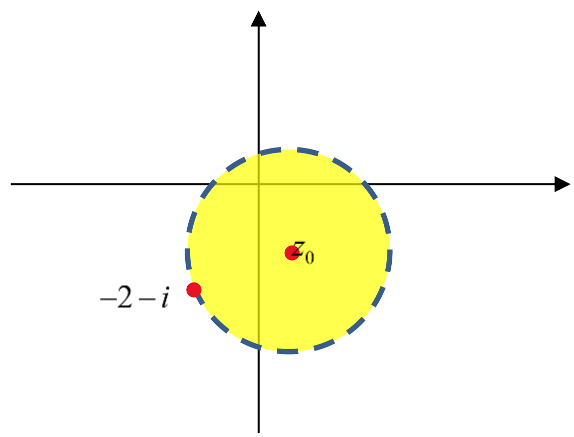


$$f(z)=\sum_{n=0}^{\infty}\frac{(-1)^n 2i}{(2+i+z_0)^{n+1}}\cdot(z-z_0)^n$$

$$\left|-\frac{z-z_0}{2+i+z_0}\right|<1$$

Converges on $|z-z_0|<|2+i+z_0|$.

4) $\frac{1}{(1-z)^3}=\sum_{n=0}^{\infty}\frac{1}{2}\frac{(n+2)(n+1)}{(1-z_0)^{n+3}}(z-z_0)^n$ Converges on $|z-z_0|<|1-z_0|$.



Laurent Series

Questions

1) Find the Laurent series expansion of the function $f(z)=\frac{1}{1-z}$ $(z\neq 1)$ around $z_0=0$, in each of the domains where the exists such an expansion.

2) Find the Laurent series expansion of the function $f(z)=\frac{1}{1-\frac{z}{2}}$ $(z\neq 2)$ around $z_0=0$, in the domain $|z|<2$ and in the domain $|z|>2$. Reminder: $f(z)=\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$.

3) Find the Laurent series expansion of the function $f(z)=\frac{1}{(z+1)(z+3)}$, $z\neq -1,-3$ around $z_0=-1$, in the domain $|z+1|>2$ and in the domain $0<|z+1|<2$.

4) Find the Laurent series expansion of the function $f(z)=\frac{1}{(z+2)(z+3)}$ $z\neq -2,-3$ around $z_0=-3$, in each of the domains where the exists such an expansion.

5) Find the Laurent series expansion of the function $f(z)=\frac{1}{(z+1)(z+3)}$, $z\neq -1,-3$ around $z_0=0$, in the domain $1<|z|<3$. Hint: use partial fractions.

6) Find the Laurent series expansion of the function $f(z)=\frac{1}{(z+1)(z+3)}$ around $z_0=0$, in the domain $|z|>3$. Hint: use partial fractions.

7) Find the Laurent series expansion of the function $f(z)=\frac{1}{(z+1)(z+3)}$ around $z_0=0$, in the domain $|z|<1$. Hint: use partial fractions.

8) Find the Laurent series expansion of the function $f(z)=\frac{1}{1+z^2}$ $z\neq \pm i$, around $z_0=0$, in the domain $|z|<1$, and find a_{-1}.

9) Find the Laurent series expansion of the function $f(z)=\frac{1}{1+z^2}$ $z\neq\pm i$,

around $z_0=i$, in the domain $0<|z-i|<2$, and find a_{-1}.

10) Find the Laurent series expansion of the function $f(z)=\frac{1}{1+z^2}$ $z\neq\pm i$,

around $z_0=i$, in the domain $|z-i|>2$, and find a_{-1}.

11) Find the Laurent series expansion of the function $f(z)=\frac{z}{(z-1)(z-4)}$ $z\neq 1,4$,

around $z_0=1$, in the domain containing $z=5$.

12) Find the Laurent series expansion of the function $f(z)=\frac{1}{(z-2)(z-3)}$ $z\neq 2,3$,

around $z_0=0$, in the domain containing $z=1-3i$.

13) Let $f(z)=\frac{a}{z-a}$, where $0<a<1$ is a real parameter.

a. Find the Laurent series for *f* around 0 in the domain $|z|>a$.

b. Prove the identity $\sum_{n=1}^{\infty}a^n\cos(n\theta)=\frac{a\cos\theta-a^2}{1-2a\cos\theta+a^2}$.

14) Let $a\in\mathbb{C}$ and let *f* be analytic on $\mathbb{C}-\{a\}$. Suppose that $\lim_{z\to a}(z-a)f(z)=0$.

Prove that *f* extends uniquely to an entire function $\tilde{f}$ and that $\tilde{f}(a)=\frac{1}{2\pi i}\oint_{|z-a|=r}\frac{f(z)}{z-a}dz$ $\forall r>0$.

15) Let $\sum_{n=-\infty}^{\infty}a_n z^n$ be the Laurent expansion of $\frac{z}{e^{z^2}-1}$, $z\neq 0$ around $z_0=0$ in the domain $0<|z|<r$.

a. What is the largest possible value of *r*?

b. Find a_n for all $n\leq 4$

Note: this exercise requires knowledge of the classification of singular points.

16) Compute $\oint\limits_{|z|=1} z^3 \cos\left(\frac{1}{z}\right) dz$ using Laurent series.

Reminder: $f(z) = \sum\limits_{n=-\infty}^{\infty} a_n (z-z_0)^n$, $a_n = \frac{1}{2\pi i} \oint\limits_{\gamma} \frac{f(z)}{(z-z_0)^{n+1}} dz$.

17) Compute $\oint\limits_{|z|=1} e^{\frac{1}{z}} dz$ using Laurent series.

Reminder: $f(z) = \sum\limits_{n=-\infty}^{\infty} a_n (z-z_0)^n$, $a_n = \frac{1}{2\pi i} \oint\limits_{\gamma} \frac{f(z)}{(z-z_0)^{n+1}} dz$.

Answer Key

1) $f(z)=\sum_{n=0}^{\infty} z^n$ for $|z|<1$, $f(z)=\sum_{n=-\infty}^{-1} -z^n$ for $|z|<1$

2) $f(z)=\sum_{n=0}^{\infty} \frac{1}{2^n} z^n$ for $|z|<2$, $f(z)=\sum_{n=-\infty}^{-1} -\frac{1}{2^n} z^n$ for $|z|>2$

3) $f(z)=\sum_{n=-1}^{\infty} \frac{(-1)^{n+1}}{2^{n+2}}(z+1)^n$ for $0<|z+1|<2$, $f(z)=\sum_{n=-\infty}^{-2} \frac{(-1)^n}{2^{n+2}}(z+1)^n$ for $|z+1|>2$

4) $f(z)=\sum_{n=-1}^{\infty} -(z+3)^n$; $0<|z+3|<1$, $f(z)=\sum_{n=-\infty}^{-2} (z+3)^n$; $|z+3|>1$

5) $f(z)=\sum_{n=-\infty}^{-1} -\frac{1}{2}(-1)^n z^n + \sum_{n=0}^{\infty} -\frac{1}{6}\frac{(-1)^n}{3^n} z^n$ $1<|z|<3$, Partial fractions: $\frac{1}{(z+1)(z+3)}=\frac{1}{2}\frac{1}{z+1}-\frac{1}{2}\frac{1}{z+3}$

6) $f(z)=\sum_{n=-\infty}^{-1} (-1)^n\left[-\frac{1}{2}+\frac{1}{6}3^{-n}\right]z^n$

7) $f(z)=\sum_{n=0}^{\infty} (-1)^n\left[\frac{1}{2}-\frac{1}{6\cdot 3^n}\right]z^n$,

8) $f(z)=\sum_{n=0}^{\infty} (-1)^n z^{2n}$, $a_{-1}=0$

9) $f(z)=\sum_{n=-1}^{\infty} (-1)^{n+1}\left(\frac{1}{2i}\right)^{n+2}(z-i)^n$, $a_{-1}=\frac{1}{2i}$

10) $f(z)=\sum_{n=-\infty}^{-2} (-1)^{-n-2}(2i)^{-n-2}(z-i)^n$, $a_{-1}=0$

11) $f(z)=(z-1)^{-1}+\sum_{n=-\infty}^{-2} \frac{4}{9}\cdot 3^{-n}(z-1)^n$, $|z-1|>3$

12) $f(z)=\sum_{n=-\infty}^{-1} (3^{-n-1}-2^{-n-1})z^n$, $|z|>3$

13) a. $f(z)=\sum_{n=-\infty}^{-1} a^{-n}z^n$ b. Proof

14) Proof

15) a. $r_{\max}=\sqrt{2\pi}$ b. $a_{-1}=1$, $a_1=-\frac{1}{2}$, $a_3=\frac{1}{12}$, All other a_n are 0.

16) $\oint_{|z|=1} z^3\cos\left(\frac{1}{z}\right)dz=\frac{\pi i}{12}$

17) $\oint_{|z|=1} e^{\frac{1}{z}}dz=2\pi i$