proprep

Workbook

Table of Contents

Conformal Mappings, and Mobius Transformations

Conformal Mappings

Questions

1) Show that the transformation $T(z)=z+1$ maps the domain $A=\{z\in\mathbb{C}:|z|<1\}$ to the domain $B=\{w\in\mathbb{C}:|w-1|<1\}$.

2) Compute $f(-6),\ f(6),\ f(6i),\ f(-6i)$ where $f(z)=e^{i\frac{\pi}{2}}\cdot z$.
What is the geometric meaning of the transformation f ?

3) Find the image of the unit circle under the transformation $f(z)=2z$.
What is the geometric meaning of the transformation f ?

4) Find a transformation f which maps the upper half-plane H^{+} onto the right half-plane H^{r}, where $H^{+}=\{z\in\mathbb{C}\mid \operatorname{Im} z>0\}$ and $H^{r}=\{z\in\mathbb{C}\mid \operatorname{Re} z>0\}$.

5) Find a transformation f which maps the 1st quadrant Q_1 onto the 3rd quadrant Q_3, where $Q_1=\{z\in\mathbb{C}\mid \operatorname{Im} z>0, \operatorname{Re} z>0\}$ and $Q_3=\{z\in\mathbb{C}\mid \operatorname{Im} z<0, \operatorname{Re} z<0\}$.

6) Find the image, B, of the set $A=\{z\in\mathbb{C}:|z|=2\}$ under the transformation $f(z)=\dfrac{1}{z}$.
This $f:\mathbb{C}-\{0\}\to\mathbb{C}-\{0\}$ is known as the *inversion mapping [transformation]*.

7) Find the image of the set $A=\left\{z=re^{i\frac{\pi}{4}}:r\geq 0\right\}$ under the transformation $f(z)=z^{2}$.

8) Find the image of the set $A=\{z=e^{i\theta}:0\leq\theta\leq\pi\}$ under the transformation $f(z)=\dfrac{1}{2}\left(z+\dfrac{1}{z}\right)$,known as the Joukowski [Zhukovsky] transformation.

9) Find the image of the set $A=\{z=x+iy:-\pi<y\leq\pi,\ 0\leq x\leq 1\}$ under the transformation $f(z)=e^{z}$.

10) Recall that the principal branch of the logarithm is defined on $\mathbb{C}-(-\infty,0]$ by $\underbrace{\text{Log}}_{\text{or Ln}}(z)=\ln r+i\theta$ where z can be expressed uniquely as $z=re^{i\theta},\ r>0,\ -\pi<\theta<\pi$.
Find the image of $\mathbb{C}-(-\infty,0]$ under the transformation $w=\text{Log}\,z$.

11) Find the image of $\overline{D(0,1)}-(-\infty,0]$ under the transformation $w=\text{Log}\,z$.
Recall: $\overline{D(0,1)}=\{z:|z|\leq 1\}$.

12) Find the image of the square $S=\{z=x+iy:0\leq x\leq 1,\ 0\leq y\leq 1\}$ under the transformation $w=z^{2}$. Where is this transformation conformal?

Answer Key

1) See in the video

2) $f(-6)=-6i$, $f(6)=6i$, $f(6i)=i\cdot 6i=-6$, $f(-6i)=i(-6i)=6$

f is a rotation around 0 by an angle of $\frac{\pi}{2}$.

3) $\{w\in\mathbb{C}:|w|=2\}$ (circle with center at 0 and radius 2)

f is a scaling with center 0 and scale factor 2.

4) $f(z)=e^{-i\frac{\pi}{2}}\cdot z$ or $f(z)=-iz$

5) $f(z)=e^{i\pi}\cdot z$ or $f(z)=-z$

6) $B=\left\{w\in\mathbb{C}:|w|=\frac{1}{2}\right\}$

7) $f(A)=\{w=\rho e^{i\frac{\pi}{2}}:\rho\geq 0\}$

8) $f(A)=[-1,1]=\{w\in\mathbb{R}:-1\leq w\leq 1\}$

9) $f(A)=\{w\in\mathbb{C}:1\leq|w|\leq e\}$

10) $\{w=u+iv:-\pi<v<\pi\}$

11) $\{w=u+iv:u\leq 0\wedge -\pi<v<\pi\}$

12) Not conformal at 0.

Mobius Transformations

Questions

1) Let f,g be Möbius Transformations: $f(z)=\dfrac{az+b}{cz+d}$, $g(z)=\dfrac{Az+B}{Cz+D}$.

show that $f\circ g$ is also a Möbius Transformation.

Compute the matrix product $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\cdot\begin{bmatrix} A & B \\ C & D \end{bmatrix}$. How is this related to the above?

2) A Möbius Transformation is of the form $f(z)=\dfrac{az+b}{cz+d}$ with $ad-bc\neq 0$.

Show that if $ad-bc=0$ then f is a constant function.

3) Find the image of $D=\{z\in\mathbb{C}:|z|<1\}$ under the mapping $w=f(z)=\dfrac{z-1}{z+1}$.

4) Find the image of $H^{+}=\{z\in\mathbb{C}\mid \operatorname{Im} z>0\}$ under the mapping $w=f(z)=\dfrac{z-1}{z+1}$.

5) Find the image of $H^{+}=\{z\in\mathbb{C}\mid \operatorname{Im} z>0\}$ under the mapping $w=f(z)=\dfrac{z+i}{z-i}$.

6) Find the image of $D=\{z\in\mathbb{C}:|z|<1\}$ under the mapping $w=f(z)=\dfrac{z+i}{z-i}$.

7) Find the image of $H^{+}=\{z\in\mathbb{C}\mid \operatorname{Im} z>0\}$ under the mapping $f(z)=\dfrac{z-a}{z-\overline{a}}$, where $a\in H^{+}$.

8) Find a conformal mapping* from the open first quadrant QI onto the open unit disk D.

$QI=\{z\in\mathbb{C}:\operatorname{Re} z>0\wedge \operatorname{Im} z>0\}$, $D=\{z\in\mathbb{C}:|z|<1\}$. *Not necessarily Möbius.

9) Let $f(z)=\dfrac{z}{z-2i}$ and let $D=\{z\in\mathbb{C}:|z-i|\leq 1\wedge \operatorname{Re} z\geq 0\}$. Find and sketch $f(D)$.

10) Find a conformal mapping from the slit open unit disk $D(0,1)-[0,1)$ onto the open unit disk D.

11) Find a conformal mapping from the strip $S=\{z\in\mathbb{C}:0<\operatorname{Im}z<1\}$ onto the open first quadrant.

12) Find a 1-1 conformal mapping from $A=\left\{re^{i\theta}:r>0\wedge 0<\theta<\frac{\pi}{4}\right\}$ onto $B=\{x+iy:0<y<1\}$.

13) Find a 1-1 conformal mapping from $A=\left\{re^{i\theta}:r>0\wedge 0<\theta<\frac{\pi}{4}\right\}$ onto $B=D(0,1)$ [unit disk].

14) Find a 1-1 conformal mapping from $A=\mathbb{C}-[0,1]$ [slit plane] onto $B=D(0,1)$ [unit disk].

15) Recall that the upper half plane is $H^{+}=\{z\in\mathbb{C}:\operatorname{Im}z>0\}$. Let $z_1,z_2\in H^{+}$.
Find a conformal equivalence* $f:H^{+}\to H^{+}$ such that $f(z_1)=z_2$.
Conformal equivalence: an analytic map which is 1-1 and onto.

Answer Key

1) $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} aA+bC & aB+bD \\ cA+bC & cB+bD \end{bmatrix}$

Matrix of composition = product of the matrices.

2) See in the video

3) $f(D) = H^l = \{z : \operatorname{Re} z < 0\}$

4) $f(H^+) = H^+$

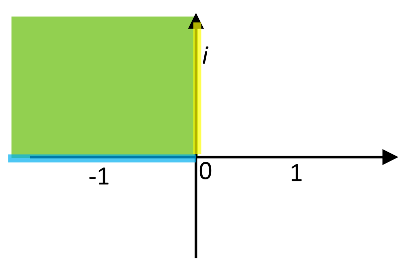
5) $f(H^+) = \{z \in \mathbb{C} : |z| > 1\}$ (Exterior of closed unit disk).

6) $f(D) = H^l = \{z \in \mathbb{C} : \operatorname{Re} z < 0\}$

7) $f(H^+) = D = \{z \in \mathbb{C} : |z| < 1\}$ (Open unit disk).

8) $z \mapsto \frac{z^2 - i}{z^2 + i}$ (Other answers are possible.)

9) $f(D) = \{z \in \mathbb{C}\mathbb{C} \mid \operatorname{Re} z \le 0,\ \operatorname{Im} z \ge 0\}$



10) See in the video

11) $f(z) = e^{\frac{\pi}{2} z}$ (Other answers are possible.)

12) $f(z) = \frac{4}{\pi} \operatorname{Log} z$ (Other answers are possible.)

13) $f(z) = \frac{z^4 - i}{z^4 + i}$ (Other answers are possible.)

14) See in rhe video

15) $f(z) = \frac{(-z_2 + \overline{z_2})z + (\overline{z_1} z_2 - z_1 \overline{z_2})}{-z_1 + \overline{z_1}}$ (Other answers are possible.)