proprep

Workbook

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Exam Review Questions

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Questions

1) Let $D = D(0,1)$. Is there an analytic $f : D \to \mathbb{C}$ such that,
$\left| f(z) \right| = \ln\left(2 + |z|\right)\ \forall z \in D$?

2) Suppose that $f(z)$ is analytic in the annulus $0 < |z| < \infty$,
i.e. the punctured plane $\mathbb{C} - \{0\}$. Suppose that $\alpha \in \mathbb{R} - \mathbb{Z}$ (real, non-integer),
such that $\forall R > 0, \int_0^{2\pi} \left| f\left(Re^{i\theta}\right) \right| d\theta \le 2\pi R^{\alpha}$ .Prove that $f(z) \equiv 0$ in the annulus.

3) Given $|a| < 1$. How many solutions does the equation $e^{z+2} = \left(\dfrac{z-a}{1-\bar{a}z} \right)^{2000}$,
have in $D(0,1)$?

4) Prove or disprove: there exists an analytic function f on $D = D(0,1)$ such that,
$f\left(\dfrac{1}{n}\right) + f\left(-\dfrac{1}{n}\right) = \dfrac{1}{n^2} + \dfrac{1}{n^3}$ for all $n \in \mathbb{N}$.

5) Show that the series $\sum_{n=0}^{\infty} \left(\dfrac{e^z - i}{e^z + i} \right)^n$ converges absolutely in the strip $\{ z \in \mathbb{C} : 0 < \operatorname{Im} z < \pi \}$.

6) Suppose $f = u + iv$ is an entire function such that $v(x,y) = \cosh u(x,y)$.
Prove that f is constant.

7) Let $R > 0$ (arbitrary). Prove that there exists $N \in \mathbb{N}$ such that, for all $n > N$,
the equation $1 + z + \dfrac{z^2}{2!} + \ldots + \dfrac{z^n}{n!} = 0$ has no solution in $D_R = D(0,R)$.

8) Prove/disprove:
There exists a sequence $(a_n)_{n=1}^{\infty}$, with $0 \ne a_n \in \mathbb{C}$, such that:
$\sum_{n=0}^{\infty} |a_n| < \infty$ and such that, for all $k \in \mathbb{N}$, $\sum_{n=0}^{\infty} \dfrac{k^n a_n}{(2k+1)^n} = 0$.

9) Prove that for all $t \in \mathbb{R}$, $\frac{1}{2\pi}\int_0^{2\pi} e^{2t\cos\theta}d\theta = \sum_{n=0}^{\infty}\frac{t^{2n}}{(n!)^2}$.

10) Let f be analytic in $D(0,1)$. Prove that there exists $n \in \mathbb{N},\ n > 1$ such that $f\left(\frac{1}{n}\right) \neq \frac{1}{n+2}$.

11) Compute the real integral $\int_0^{\infty} f(x)dx$, where $f(x) = \frac{1}{x^3+1}$ $(0 \le x < \infty)$.

12) Prove or disprove:

There exists an analytic function f on $D = D(0,1)$ such that $\left|z^2 f(z) + e^z\right| \le 1\ \forall z \in D$.

13) Let $f(z)$ be analytic in the annulus $0 < |z| < 2$ and suppose that for all $n \ge 0,\ \oint_{|z|=1} z^n f(z)dz = 0$. Prove that $\lim_{z\to 0} f(z)$ exists and is finite.

14) Prove or disprove:

There exists an entire function f such that $f\left(\frac{1}{n}\right) = \frac{(-1)^n}{n}$ for all natural n.

15) Prove that there's no entire function $f(z)$ such that if $z = x$ is real then $f(x) = \begin{cases} x^4;\ x > 0 \\ -x^4;\ x < 0 \end{cases}$.

16) $\operatorname{Log} z$ is the principal branch of the log on $D = \mathbb{C} - (-\infty, 0]$.

a. Show that $\lim_{z\to 0}\left|\operatorname{Log} z\right| = \infty$

b. Show that $\lim_{z\to 0} z\operatorname{Log} z = 0$

c. Show that $\lim_{z\to z_0} z\operatorname{Log} z$ doesn't exist for any $z_0 \in (-\infty, 0)$.

17) Compute the real improper integral $\int_0^{\infty}\frac{\sqrt{x}}{1+x^2}dx$.

18) Find the Laurent Expansion of the function $f(z) = \frac{1}{(z^2+1)^4}$ in the annulus $0 < |z-i| < 2$.

19) Let $f(z)$ be analytic in the annulus $0 < |z| < 1$ and even. Show that $\oint_{|z|=\frac{1}{2}} f(z)dz = 0$.

20) $f(z) = f(-z)$. Let f be analytic on $D = D(0,2)$, and 1-1 on $U = \{z : 1 < |z| < 2\}$.

Prove that f is 1-1 on D.

Hint: use the geometric meaning of the Argument Principle.

21) Let f be analytic in $D=D(0,1)$. Prove that $\exists n\in\mathbb{N}$ such that $f\left(\frac{1}{n}\right)\cdot f\left(\frac{1}{n+1}\right)\neq\frac{1}{n}$.

22) Find the number of zeros of $h(z)=z^2-4-e^{-3z}$ in $H^r=\{z:\operatorname{Re}z>0\}$.
Hint: use Rouché's Theorem.

23) Define $f(z)=\dfrac{(1+z^2)-i(1-z^2)}{(1+z^2)+i(1-z^2)}$.
Let $A=\{z:\operatorname{Re}z>0,\operatorname{Im}z>0\}$, the open 1st quadrant. Find $f(A)$, the image of A under f.

24) Let $n\in\mathbb{N}$ and let $P(z)=1z^n+a_{n-1}z^{n-1}+\dots a_1z+a_0$ be a (normalized) polynomial.
Prove that all the roots of the polynomial are in the circle $|z|<\sqrt{1+|a_{n-1}|^2+\dots+|a_1|^2+|a_0|^2}$
Hint: use the Cauchy-Schwarz inequality and Rouché's theorem.

25) Let $f:D(0,1)\to\mathbb{C}$ [not necessarily continuous] be such that f^2, f^3 are both analytic.
Prove that f is analytic.

26) Define $f:\mathbb{C}\to\mathbb{C}$ as follows. $f(0)=0$ and if $z\neq 0$ then $z=re^{i\theta}$ uniquely, with $-\pi<\theta\leq\pi$,
and we define $f(z)=\sqrt{r}e^{i\frac{\theta}{2}}$. Prove that f^2 is analytic but f is not even continuous.

27) Compute the integral $\oint_{|z|=1}\frac{1}{z^2}\cdot\frac{1}{e^{\frac{1}{z}}-1}dz$. Hint: residue at infinity.

28) Compute the integral $\oint_{|z|=1}\dfrac{\frac{1}{z^2}\cos e^{\frac{1}{z}}}{z-2}dz$.

29) Let $A=\left\{z\in\mathbb{C}:-\frac{\pi}{4}<\operatorname{Re}z<\frac{\pi}{4}\right\}$ and define $f:A\to\mathbb{C}$ by $f(z)=\tan z$, Find $f[A]$.

z

30) Let $f=u+iv$ be analytic on $D=D(0,1)$. $f(x+iy)=u(x,y)+iv(x,y)$.
Suppose we are given that $|u(x,y)|+|v(x,y)|=1$ for all $z\in D$.
Prove that f is constant. Hint: use the open mapping theorem.

Answer Key

All questions here are proofs, so there are no “answers”.