proprep

Workbook

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Sequences and Topology

Sequences of Complex Numbers

Questions

1) $z_n = \frac{1}{n} + i\left(\frac{n-2}{n}\right)$, $\lim_{n\to\infty} z_n = ?$

2) $z_n = \frac{n^2+n+1}{3n^2+2} + i\left(\frac{n-1}{2n}\right)$, $\lim_{n\to\infty} z_n = ?$

3) Given: $z_n = i^{2n}\cdot n^3$. Prove: $\lim_{n\to\infty}|z_n| = \infty$ and use this to evaluate $\lim_{n\to\infty} z_n$.

4) Given: $z_n = \frac{i^n}{n}$. Prove: $\lim_{n\to\infty}|z_n| = 0$ and use this to evaluate $\lim_{n\to\infty} z_n$.

5) Given: $z_n = \frac{(1+i)^n}{n}$. Prove: $\lim_{n\to\infty}|z_n| = \infty$ and use this to evaluate $\lim_{n\to\infty} z_n$.

6) Given a sequence $z_n = n\cdot z^n$ where $z \in \mathbb{C}$ is fixed. Find $\lim_{n\to\infty}|z_n|$ and $\lim_{n\to\infty} z_n$ for two cases:

a. $|z| \geq 1$

b. $0 < |z| < 1$

7) Prove: If $z_n \xrightarrow{n\to\infty} z$ and $w_n \xrightarrow{n\to\infty} = w$, then $z_n + w_n \xrightarrow{n\to\infty} z + w$.

8) For each sequence, check if it converges; if so, compute its limit:

c. $z_n = \frac{1+n}{1-2n} + \frac{n-10}{n^2}i$

d. $z_n = \cos(\pi n) + n\sin\left(\frac{1}{n}\right)i$

e. $z_n = \left(1+\frac{2}{n}\right)^{-n} + \sqrt[n]{3^n+4^n}\cdot i$

f. $z_n = \left(\frac{1+\sqrt{3}i}{2}\right)^n$

Answer Key

1) $\lim_{n\to\infty} z_n = i$

2) $\lim_{n\to\infty} z_n = \frac{1}{3} + i\frac{1}{2}$

3) $\lim_{n\to\infty} z_n = \infty$

4) $\lim_{n\to\infty} z_n = 0$

5) $\lim_{n\to\infty} z_n = \infty$

6) a. $\lim_{n\to\infty} z_n = \infty$ b. $\lim_{n\to\infty} z_n = 0$

7) Proof

8) a. Converges to $-\frac{1}{2}$ b. Doesn't converge c. Converges to $\frac{1}{e^2} + 4i$ d. Doesn't converge

Basic Topological Concepts

Questions

1) Sketch the set *S* defined by the inequality $|z-i|+|z+i|<4$.

a. Is *S* open?
b. Is *S* a domain?
c. Is *S* a simply-connected domain

2) Sketch the set *S* defined by the equation $\mathrm{Re}\left[\frac{z-a}{z+a}\right]=0$, where $a\neq 0$ is real.

a. Is *S* open?
b. Is *S* a domain?
c. Is *S* a simply-connected domain?

3) Sketch the set *S* defined by the equation $\mathrm{Im}\left[\frac{z-1}{z+1}\right]=0$.

a. Is *S* open?
b. Is *S* a domain?
c. Is *S* a simply-connected domain?

4) Sketch the set *S* defined by the equation $|z+i|=2|z-i|$.

a. Is *S* open?
b. Is *S* a domain?
c. Is *S* a simply-connected domain?

Answer Key

1) a. Yes b. Yes c. Yes

2) a. *S* is not open.
b. Hence S is not a domain
c. Hence S is not a simply-connected domain

3) a. *S* is not open.
b. Hence S is not a domain
c. Hence S is not a simply-connected domain

4) a. *S* is not open.
b. Hence S is not a domain
c. Hence S is not a simply-connected domain

Appolonian Circles

Questions

1) a. Show that for any $z, w \in \mathbb{C}$, $|z \pm w|^2 = |z|^2 \pm 2\,\mathrm{Re}\left(z \cdot \overline{w}\right) + |w|^2$.

b. Let $z_1, z_2 \in \mathbb{C}$ and $k > 0$ (real).

Show that the equation $|z - z_1| = k \cdot |z - z_2|$

describes a line $(k = 1)$ or a circle $(k \neq 1)$.

*Proof - for the solution Proof go watch the video