proprep

Workbook

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Singular Points

Zeros of Analytic Functions

Questions

1) Find the order of the zero of $f(z) = z\sin z$ at $z_0 = 0$.

2) Find the order of the zero of $f(z) = z\sin z^3$ at $z_0 = 0$.

3) Suppose that $f_1(z)$ is analytic at z_0 and has a zero of order m_1 there and suppose, too, that $f_2(z)$ is analytic at z_0 and has a zero of order m_2 there.
Prove that $f(z) = f_1(z)f_2(z)$ is analytic at z_0 and has a zero of order $m = m_1 + m_2$ there.

4) Find the order of the zero of $f(z) = z^{20}\sin z$ at $z_0 = 0$.

5) Find the order *m* of the zero of $f(z) = e^{\sin z} - \sin^2 z - 1$ at $z_0 = 0$.

6) Suppose that $f_1(z)$ is analytic at z_0 and has a zero of order 7 there and suppose, too, that $f_2(z)$ is analytic at z_0 and has a zero of order 3 there.
Show that z_0 is a zero of $f(z) = f_1(z) + f_2(z)$ and find its order.

7) Find the order *m* of the zero of $f(z) = 6\sin z^3 + z^{12}(z^6 - 6)$ at $z_0 = 0$.

8) Let $f(z)$ be continuous on the unit disk $D = D(0,1)$ and assume $f(z) \not\equiv 0$.
Suppose that $f^3(z),\ f^7(z)$ are both holomorphic on *D*. Prove that *f* is holomorphic on D.

9) Prove: there is no holomorphic function $f(z)$ on the unit disk $D = D(0,1)$,
such that $f\left(\frac{1}{n}\right) = \frac{1}{\sqrt{n}}$ for $n = 2, 3, 4, \ldots$

Answer Key

1) Order 2
2) Order 4
3) Proof
4) Order 21
5) $m=1$
6) Order 3
7) $m=3$
8) Proof
9) Proof

Classifying Singular Points

Questions

1) Find and classify the singularities of $f(z)=\frac{1}{1-z}$.

2) Suppose $h(z)=\frac{f(z)}{g(z)}$ where $f(z),\ g(z)$ are analytic around $z=0$.

Suppose, too, that $z=0$ is a zero of *f* of order 7 and of *g* of order 11.
Classify the singularity of *h* at $z=0$.

3) Suppose $h(z)=\frac{f(z)}{g(z)}$ where $f(z),\ g(z)$ are analytic around z_0.

Suppose, too, that z_0 is a zero of *f* of order *n* and of *g* of order *m*.

What kind of singularity does *h* have at z_0? Distinguish the cases $n\geq m$ and $n<m$.

4) Classify the singularity of $f(z)=\frac{1-\cos z}{z^5}$ at $z_0=0$.

5) Find and classify the singularities of $f(z)=ze^{\frac{1}{z}}$.

6) Find and classify the singularities of $f(z)=\frac{1}{e^{-z}-1}+\frac{1}{z}$.

7) Find and classify the singularities of $f(z)=\frac{\sin z}{\cos z}=\tan z$.

8) Find and classify the singularities of $f(z)=\frac{1}{z^2-1}\cos\left(\frac{\pi z}{z+1}\right)$.

9) Find and classify the singularities of $f(z)=e^{\frac{z}{z-2}}$.

10) Find and classify the singularities of $f(z)=\frac{1}{\sin\left(\frac{1}{z}\right)}$.

11) Find and classify the singularities of $f(z)=z\cot z$.

12) Suppose that f is analytic in $\mathbb{C}-\{0\}$, that 0 is a pole of f of order *m* and that $-1\notin f\left[\mathbb{C}-\{0\}\right]$.

Find and classify the singularities of $g(z)=\dfrac{f(z)-1}{f(z)+1}$.

13) Find all zeros and poles, and their orders, of $f(z)=\dfrac{(z-2)^2}{\left(e^{2z}-1\right)^2 z^3}$ in the domain $|z|<4$.

14) Let $f(z)=\dfrac{\tan z}{z^2-\frac{\pi}{4}z}$. Classify the singularities of *f* at $z=0$ and at $z=\dfrac{\pi}{4}$.

15) Let $f,g:\mathbb{C}\to\mathbb{C}$ be two entire functions which are not constant, and suppose that $|f(z)|\le|g(z)|\ \forall z\in\mathbb{C}$. Prove that:

a. All the singularities of $h(z)\equiv\dfrac{f(z)}{g(z)}$ are isolated and removable.

b. $f(z)=c\cdot g(z)$ for some constant *c* satisfying $|c|\le 1$

16) Find and classify the singularities of $f(z)=\dfrac{z^2-\frac{1}{4}}{\sin\frac{\pi}{z}}$.

17) Show that the point $z=i$ is an essential singularity of the function $f(z)=\cos\left(\dfrac{1}{z^2+1}\right)$

18) Let $f:D\to\mathbb{C}$ be analytic in the domain $D=\{z\,|\,0<|z-z_0|<1\}$ and let $a\in\mathbb{R}$ be a constant such that $0\le a<1$. Suppose that $\left|(z-z_0)^a f(z)\right|\le 1\ \forall z\in D$.

Prove that z_0 is a removable singularity of *f*.

19) Let $f:D\to\mathbb{C}$ be analytic in the domain $D=\{z\,|\,0<|z|<1\}$ and suppose

$$f\left(\frac{1}{n}\right)=\frac{1}{n!}\quad\forall n\in\mathbb{N}.$$

Prove that the point $z=0$ is an essential singularity of *f*.

20) Suppose that z_0 is a pole of $f(z)$ [of order *m*, say] and let $F(z)=e^{f(z)}$.

Prove that z_0 is an essential singularity of $F(z)$.

Answer Key

1) simple pole at $z=1$
2) pole of order 4
3) $n \geq m$: removable singularity , $n < m$: pole of order $m-n$
4) pole of *f* of order 3
5) 0 is an essential singularity
6) $z_0 = 0$: removable singularity , $z_k = 2\pi ik,\ 0 \neq k \in \mathbb{Z}$: simple poles
7) $z_k = \frac{\pi}{2} + \pi k,\ k \in \mathbb{Z}$: simple poles
8) $z=1$: removable singularity , $z=-1$: essential singularity
9) $z=2$: essential singularity
10) $z_k = \frac{1}{\pi k}\ 0 \neq k \in \mathbb{Z}$: simple poles , $z_0 = 0$: non-isolated singularity
11) $z_0 = 0$: removable singularity , $z_k = \pi k, k \neq 0$: simple poles
12) $z=0$: removable singularity
13) $z=2$ is a zero of order 2 , $z=0$ is a pole of order 5 , $z=\pm \pi i$ are poles of order 2
14) $z=0$: removable singularity , $z=\frac{\pi}{4}$: simple pole
15) Proof
16) $z=\frac{1}{k},\ k=\pm 2$: removable singularity , $z=\frac{1}{k},\ k \neq \pm 2$: simple pole $z=0$: non-isolated singularity
17) Proof
18) Proof
19) Proof
20) Proof

Singular Points at Infinity

Questions

1) Classify the singularity at ∞ of $f(z)=\frac{z^2}{1+z}$.

2) Classify the singularity at ∞ of $f(z)=e^z$.

Answer Key

1) Pole of order 1 (simple pole).

2) Essential singularity.

Casorati–Weierstrass Theorem

Questions

1) a. Prove that $z=i$ is an essential singularity of $f(z)=\cos\left(\frac{1}{z^2+1}\right)$.

b. Deduce that there exists $z\in\mathbb{C}$ such that $\left|\cos\left(\frac{1}{z^2+1}\right)+100\tan^2 z+e^{-z^2}-5i\right|<1$

*Proof- for the full solution see the video