proprep

Workbook

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Eigenvectors Eigenvalues and Diagonalization

Eigenvectors and Eigenvalues of a Matrix

Questions

1) Find the eigenvalues and eigenvectors of the matrix $A=\begin{pmatrix} 4 & -1 & -1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$.

2) Find the eigenvalues and eigenvectors of the matrix $A=\begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}$.

3) Find the eigenvalues and eigenvectors of the matrix $A=\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix}$.

4) Find the eigenvalues and eigenvectors of the matrix $A=\begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$.

5) Find the eigenvalues and eigenvectors of the matrix $A=\begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$.

6) Given the matrix $A=\begin{bmatrix} k-2 & 2k & k+1 \\ k-1 & -1 & 2 \\ -k & 0 & -6 \end{bmatrix}$.

For which value(s) of the parameter *k* will the value 2 be an eigenvalue of *A*?

7) Given a square matrix A.
True or false? Prove your answer:

a. 0 is an eigenvalue of A if and only if A is non-invertible.

b. If A is invertible and λ is an eigenvalue of A then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

c. A and A^T have the same characteristic polynomial.

d. A and A^T have the same eigenvectors.

8) Given two square matrices A and B.
True or false? Prove your answer:

a. AB and BA have the same eigenvalues.

b. If v is a nonzero eigenvector of both A and of B, then v is also an eigenvector of $4A+10B$.

9) Let $W \subseteq M_{n\times n}$ be the subset of $n \times n$ matrices for which some v is an eigenvector.

a. Prove that W is a vector subspace of $M_{n\times n}$.

b. For the case $n=2$, $v=\begin{bmatrix}4\\0\end{bmatrix}$, find a basis of W. What is $\dim(W)$?

10) a. Suppose that a square matrix A has an eigenvector v for the eigenvalue 4.
Define another square matrix $B = A^4 - 2A^2 + 10A - 4I$. Prove that v is also an eigenvector of B and find the corresponding eigenvalue.
b. Suppose that a square matrix A has an eigenvector v for the eigenvalue λ.
Let $p(x)$ be a polynomial and define $B = p(A)$. Prove that v is also an eigenvector of B and find the corresponding eigenvalue.

11) Let A be the matrix $\begin{bmatrix}1 & a\\4 & 1\end{bmatrix}$, where $a \in \mathbb{R}$ is a parameter.

a. For $a=3$, give an example of a vector $\begin{bmatrix}x\\y\end{bmatrix}$ which is not an eigenvector of A.

b. For which value(s) of a is $\begin{bmatrix}1\\2\end{bmatrix}$ an eigenvector of A.

c. Given any $A_{2\times 2} \neq 0$ and $0 \neq u \in \mathbb{R}^2$ which is not an eigenvector of A.
Prove that $\{v, Av\}$ is a basis of $\mathbb{R}^2$.

12) Let A, B be two $n \times n$ matrices such that $AB = BA$ and $\operatorname{rank}(A) = n-1$.

Suppose $0 \neq v \in \mathbb{R}^n$ is an eigenvector of A for the eigenvalue 0.

Prove that v is an eigenvector of B.

13) a. Let A be a square matrix of order 2.

i. Prove that the characteristic polynomial of A is $p_A(x) = x^2 - \operatorname{tr}(A)x + |A|$.

ii. Given that $\operatorname{tr}(A) = 4$ and that A has only one eigenvalue. Compute $|A|$.

b. Let A be a square matrix of order n with characteristic polynomial $p_A(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1 x^1 + a_0 x^0$. Prove: $a_{n-1} = -\operatorname{tr}(A)$ and $a_0 = (-1)^n |A|$.

14) Let B be a square matrix of order 4 and suppose that $\operatorname{rank}(B) = 1$. Prove:

a. 0 is an eigenvalue of B.
b. The geometric multiplicity of the eigenvalue 0 is 3.
c. The algebraic multiplicity of the eigenvalue 0 is either 3 or 4.
d. B has at most 2 eigenvalues.
e. If $\lambda \neq 0$ is an eigenvalue of B then $\lambda = \operatorname{tr}(B)$.

15) Let B be a square matrix of order n and suppose that $\operatorname{rank}(B) = k < n$.

a. Prove that 0 is an eigenvalue of B.
b. Prove that the geometric multiplicity of the eigenvalue 0 is $n-k$.
c. What values are possible for the algebraic multiplicity of the eigenvalue 0?

Answer Key

To view the answers to thous exercises, please refer to thr appropriate videos on site.

Matrix Diagonalization

Questions

1) Given the matrix $A=\begin{bmatrix} 0 & 2 & -1 \\ 0 & 2 & -1 \\ 0 & 1 & 0 \end{bmatrix}$.

a. Find the characteristic matrix.
b. Find the characteristic polynomial.
c. Find the eigenvalues and algebraic multiplicity of each.
d. For each eigenvalue, find its eigenspace and its geometric multiplicity.
e. Find the eigenvectors.
f. Determine if the matrix is diagonalizable.
i. Find the minimal polynomial for *A*.
j. Determine if *A* is invertible by examining its eigenvalues.
If *A* is invertible, find A^{-1} in terms of A and I, with the help of the Cayley-Hamilton theorem.

2) Given the matrix $A=\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

a. Find the characteristic matrix.
b. Find the characteristic polynomial.
c. Find the eigenvalues and algebraic multiplicity of each.
d. For each eigenvalue, find its eigenspace and its geometric multiplicity.
e. Find the eigenvectors.
f. Determine if the matrix is diagonalizable.
i. Find the minimal polynomial for *A*.
j. Determine if *A* is invertible by examining its eigenvalues.
If *A* is invertible, find A^{-1} in terms of A and I, with the help of the Cayley-Hamilton theorem.

3) Given the matrix $A=\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 1\end{bmatrix}$.

a. Find the characteristic matrix.
b. Find the characteristic polynomial.
c. Find the eigenvalues and algebraic multiplicity of each.
d. For each eigenvalue, find its eigenspace and its geometric multiplicity.
e. Find the eigenvectors.
f. Determine if the matrix is diagonalizable.
g. Diagonalize *A*. [We showed earlier that *A* is diagonalizable.]
h. Compute A^{2017} (using the above).
i. Find the minimal polynomial for *A*.
j. Determine if *A* is invertible by examining its eigenvalues.

4) Given the matrix $A=\begin{bmatrix}-1 & 3 & 0\\ 3 & -1 & 0\\ -2 & -2 & 6\end{bmatrix}$.

a. Find the characteristic matrix.
b. Find the characteristic polynomial.
c. Find the eigenvalues and algebraic multiplicity of each.
d. For each eigenvalue, find its eigenspace and its geometric multiplicity.
e. Find the eigenvectors.
f. Determine if the matrix is diagonalizable.
g. Diagonalize *A*. [We showed earlier that *A* is diagonalizable.]
h. Compute A^{2017} (using the above).
i. Find the minimal polynomial for *A*.
j. Determine if *A* is invertible by examining its eigenvalues.
If *A* is invertible, find A^{-1} in terms of A and I, with the help of the Cayley-Hamilton theorem.

5) Given the matrix $A=\begin{bmatrix}3 & -2\\ 4 & -1\end{bmatrix}$.

Find the eigenvalues and corresponding eigenvectors of *A*. If *A* is diagonalizable, find an invertible matrix *P* such that $P^{-1}AP=D$ where *D* is a diagonal matrix.
Solve twice: once over $\mathbb{R}$ and once over $\mathbb{C}$.

6) Given the matrix $A=\begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}$.

Find the eigenvalues and corresponding eigenvectors of *A*. If *A* is diagonalizable, find an invertible matrix *P* such that $P^{-1}AP=D$ where *D* is a diagonal matrix.
Solve twice: once over $\mathbb{R}$ and once over $\mathbb{C}$.

7) Given the matrix $A=\begin{bmatrix} a & b & b \\ -1 & 3 & 2 \\ 2 & -8 & -5 \end{bmatrix}$.

a. For which value(s) of *a* and *b* will the eigenvalues of *A* be 1 and -1 (only)?
b. Using the values *a* and *b* found above, determine if *A* is diagonalizable.

8) Let *A* be a real matrix of order 3×3. Given that the eigenvectors of *A* are

$v_1=\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $v_2=\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $v_3=\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and that they correspond to eigenvalues $\lambda_1=6$, $\lambda_2=2$, $\lambda_3=-4$.

Find the matrix *A*.

9) Determine if there exists a 3×3 matrix with eigenvectors $v_1=\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $v_2=\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, $v_3=\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$

corresponding to eigenvalues $\lambda_1=1$, $\lambda_2=2$, $\lambda_3=3$. If such a matrix exists, find it.

10) True or false? Prove your answer:
a. Every diagonalizable matrix is invertible.
b. Every diagonalizable matrix is not invertible.
c. Every matrix is diagonalizable.
d. There exists a matrix *A* with an eigenvector $\begin{bmatrix} 4 \\ 1 \\ 10 \end{bmatrix}$ corresponding to an eigenvalue of 14.

11) Let *A* be a diagonalizable square matrix.
a. Prove that for any scalar *k*, the matrix $A+kI$ is also diagonalizable.
b. If 4 is an eigenvalue of *A*, find an eigenvalue of $A+kI$.

12) Let *A* be a real 3×3 matrix. Suppose that v_1, v_2 are eigenvectors of *A* for eigenvalue $\lambda=1$ and that v_3 is an eigenvector of *A* for eigenvalue $\lambda=-1$.

True or false? Prove your answer:

a. v_3 is a linear combination of the vectors v_1, v_2.

b. If the vectors v_1, v_2 are linearly independent, then $A^{2018}=I$.

c. *A* is diagonalizable.

13) Given two $n\times n$ matrices as follows:

A diagonalizable matrix B and an invertible matrix Q.

True or false? Prove your answer:

a. $Q^{-1}BQ$ is a diagonal matrix.

b. $Q^{-1}BQ$ is diagonalizable.

14) Let $A_{3\times3}$ be a real matrix of the form $A=\begin{bmatrix} a & b & c \\ 4a & 4b & 4c \\ 10a & 10b & 10c \end{bmatrix}$.

Suppose A has a nonzero eigenvalue. Prove that A is diagonalisable.

15) Let $A_{1\times n}$ be a $1\times n$ row-matrix $\begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix}$, where $n>1$. Let $B=A^T A$.

Show that B is a square matrix and find its eigenvalues.

16) Let $A\in M_{5\times5}[\mathbb{R}]$, i.e, A is a real square matrix of order 5.

Prove or disprove [True or False] the following:

a. There exists $\lambda\in\mathbb{R}$ such that the λ-eigenspace $W_\lambda=\{v\in\mathbb{R}^5 : Av=\lambda v\}$ is a nontrivial* subspace of $\mathbb{R}^5$. *In other words, $\dim W_\lambda>0$ or $W_\lambda\neq\{0\}$.

b. If $v_1, v_2\in\mathbb{R}^5$ are eigenvectors of A, then so is v_1+v_2.

c. If A, B are row equivalent square matrices then they have the same eigenvalues.

d. If $A\in M_{n\times n}[\mathbb{R}]$ and A is diagonalisable then all its eigenvalues are distinct.

e. If $A\in M_{n\times n}[\mathbb{R}]$ and all its eigenvalues are distinct then A is diagonalisable over $\mathbb{R}$.

17) Let $A \in M_{4\times4}[\mathbb{R}]$. Given: A has 4 real eigenvalues, the smallest being 2 and the greatest 4. Which of the following are True/False. Prove/Disprove:

a. $\text{rank}\, A = 4$.

b. A is diagonalisable .

c. $\text{tr}\, A > 10$.

d. $|A| < 127$.

e. There exists an eigenvector v of A such that $A^2 v = 2v$.

18) Let A be a square matrix of order $n > 1$ over $\mathbb{R}$ or $\mathbb{C}$.
True or False [prove or disprove]:

a. If v is an eigenvector of A then v is also an eigenvector of A^n.

b. If v is an eigenvector of A^n then v is also an eigenvector of A.

c. If A is diagonalisable then so is A^n.

d. If A^n is diagonalisable then so is A.

Answer Key

To view the answers to thous exercises, please refer to thr appropriate videos on site.

Cayley-Hamilton theorem and the Minimal Polynomial

Questions

1) Let A be a square $n \times n$ matrix which is **idempotent**, i.e. $A^2 = A$.

a. Prove that each eigenvalue of A is either 0 or 1.
b. List all possibilities for the minimal polynomial of A.
c. Prove that the characteristic polynomial of A can be factored into linear factors.
d. Prove that A is diagonalisable.
e. Prove that $\text{tr}(A) = \text{rank}(A)$.

2) Let $A \in M_{3\times3}[\mathbb{R}]$. Given: $\text{tr}(A) = 0,\ |A| = 0$ and $\lambda = 1$ is an eigenvalue of A.

Show that A is diagonalisable and find all its eigenvalues.

3) Let $A \in M_{3\times3}[\mathbb{R}]$. Given: $0 < \text{rank}(A - 10I) < \text{rank}(A - 4I) < 3$.

a. Find the eigenvalues of A and their geometric multiplicity.
b. Find the algebraic multiplicity of the eigenvalues and the characteristic polynomial of A.
c. Determine if A is invertible.
d. Determine if A is diagonalisable.

4) Let A be a square matrix of order $n \geq 2$ whose minimal polynomial is $m_A(x) = (x-1)^2$.

Prove that the matrix $B = A^2 + 4A + 3I$ is invertible.

Answer Key

To view the answers to thous exercises, please refer to thr appropriate videos on site.

Matrix Similarity

Questions

1) a. Define the concept of similarity of matrices.

b. Given that A and B are similar matrices, prove:

i. $|A| = |B|$.

ii. $tr(A) = tr(B)$.

iii. A and B have the same characteristic polynomial.

2) Prove that if $B = P^{-1}AP$, then $B^n = P^{-1}A^nP$, $n = 1, 2, 3, 4, \ldots$

3) Answer the following Questions

a. Let A be a real $n \times n$ matrix an suppose that A is similar to 4*A*.
Prove that A is not invertible.

b. Prove that the matrices $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $4A = \begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix}$ are similar.

4) Given the real matrix $A = \begin{bmatrix} a & b & b \\ -1 & 3 & 2 \\ 2 & -8 & -5 \end{bmatrix}$.

Do there exist real constants a, b such that A is similar to $B = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 7 & 0 \\ 9 & -17 & -6 \end{bmatrix}$?

5) Given three square matrices of order n: A, B, C.

a. Prove: A is similar to itself.

b. Prove: if A is similar to B then B is similar to A.

c. Prove: if A is similar to B and B is similar to C then A is similar to C.

d. Prove: if A is similar to B and both are invertible then A^{-1} is similar to B^{-1}.

e. Prove: if A is similar to B then A^k is similar to B^k for any natural k.

f. Prove: if A is similar to B and $q(x)$ is a polynomial then $q(A)$ is similar to $q(B)$.

g. Prove: if A is similar to B then A^T is similar to B^T.

Let A and B be similar $n \times n$ matrices.

h. Prove: $\text{rank}(A) = \text{rank}(B)$

i. Prove: $\text{null}(A) = \text{null}(B)$

Hint: $\text{rank}(AB) \le \text{rank}(B)$, $\text{rank}(AB) \le \text{rank}(A)$

6) True/False [Prove/Disprove] the following. Assume all matrices are over $\mathbb{R}$.

a. If two 3×3 matrices have the same characteristic polynomial then they are similar.

b. If two 3×3 matrices have the same minimal polynomial then they are similar.

c. If two square matrices have the same characteristic polynomial and the same minimal polynomial then they are similar.

d. The following matrices are similar: $A=\begin{bmatrix}3&0&0\\0&0&0\\0&0&0\end{bmatrix}$ $B=\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$.

Hint: two 3×3 matrices are similar if and only if they have the same characteristic and minimal polynomials.

7) Given $A_{3\times3}$ over $\mathbb{R}$ with 3 eigenvalues $\{0, 1, 2\}$.

Compute each of the following or explain why it can't be done.

a. $\text{rank}(A)$

b. $\dim(\ker(A)) = \text{nullity}(A)$

c. $\text{tr}(A)$

d. $|A^T A|$

e. the eigenvalues of $A^T A$

f. the eigenvalues of $\left(4A^2+10A+I\right)^{-1}$

8) Let A, B be similar matrices over $\mathbb{R}$. Prove that they have the same minimal polynomial.

Answer Key

To view the answers to thous exercises, please refer to thr appropriate videos on site.

Linear Transformation's Eigenvalues and Diagonalization

Questions

1) If $P \in M_2[\mathbb{R}]$, we can define a linear transformation $T_P : M_2[\mathbb{R}] \to M_2[\mathbb{R}]$ by $T_P(X) = PX$. Check that T_P really is linear.

a. Let $W \subseteq M_2[\mathbb{R}]$ consist of all P for which $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is an eigenvector of T_P. Find W.

b. Prove that W is a subspace of $M_2[\mathbb{R}]$ and find a basis for it.

2) Given the linear transformation $T : M_{10}[\mathbb{R}] \to M_{10}[\mathbb{R}],\ T(X) = PX$ where $P \in M_{10}[\mathbb{R}]$. Suppose $A \in M_{10}[\mathbb{R}]$ is invertible, and is an eigenvector of T with eigenvalue 4. Compute $|P| = \det(P)$.

3) Find a linear transformation $T : M_{2x3}(\mathbb{R}) \to M_{2x3}(\mathbb{R})$ such that $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ is an eigenvector corresponding to eigenvalue 4.

4) i. Given the matrix $A = \begin{bmatrix} 4 & -1 & -1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ over $\mathbb{R}$. Find the eigenvalues and eigenvectors of A.

ii. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation described by
$T[x, y, z] = [4x - y - z, x + 2y - z, x - y + 2z]$

a. Find the eigenvalues of T.
b. For each eigenvalue find a basis of its eigenspace.
c. Is T diagonalisable?

5) i. Let A be the matrix $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ over $\mathbb{R}$.

a. Find the characteristic matrix of A.
b. Find the characteristic polynomial of A.
c. Find the eigenvalues and the algebraic multiplicity of each.
d. Find the eigenspaces and the geometric multiplicity of each eigenvalue.
e. Find a set of eigenvectors for A.
f. Show that A is diagonalisable.
g. Diagonalise A.
h. Compute A^{2021}.
i. Find the minimal polynomial of A.
j. Determine if A is invertible by examining its eigenvalues.

ii. Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation described by $T[x,y,z]=[x+z,y,x+z]$.

a. Find the eigenvalues/eigenvectors of T.
b. Show that T diagonalisable.
c. Compute $T^{2021}[x,y,z]$.

6) Consider the linear transformation $T: M_2[\mathbb{R}] \to M_2[\mathbb{R}]$ given by $T\begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x & y \\ z & t \end{bmatrix}$.

a. Find $[T]_E$, the representation matrix of T in the standard basis E of $M_2[\mathbb{R}]$.
b. Find the eigenvalues/eigenvectors of T.
c. Show that T diagonalisable.
d. Compute $T^{10}\begin{bmatrix} x & y \\ z & t \end{bmatrix}$.

7) Consider the linear transformation $T: P_2[\mathbb{R}] \to P_2[\mathbb{R}]$ given by $T\left(p(x)\right) = p(x+1)$.

a. Find $[T]_E$, the representation of T in the standard basis E of $P_2[\mathbb{R}]$.
b. Find the eigenvalues/eigenvectors of T.
c. Show that T is not diagonalisable.

8) Let V be a vector space of dimension n over a field F and let $T:V\to V$ be a linear transformation.

a. Prove that T is invertible if and only if all its eigenvalues are nonzero.

b. Prove that if T is invertible then T^{-1} has the same eigenvectors as T.
How are the eigenvalues of T^{-1} related to the eigenvalues of T?

Answer Key

To view the answers to thous exercises, please refer to thr appropriate videos on site.