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Workbook

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Fields

Fields

Questions

1) Define addition $\oplus$ and multiplication $\otimes$ on $\mathbb{R}$ as follows:

a. $x \oplus y = x + y + 4$
$x \otimes y = 2xy$

b. $x \oplus y = x + y$
$x \otimes y = 2xy$

c. $x \oplus y = y$
$x \otimes y = y^2$

Check whether or not $F = (\mathbb{R}, \oplus, \otimes)$ satisfies each of the field axioms.

2) Let $C = (\mathbb{R}^2, +, \cdot)$ where $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$ and $+, \cdot$ are as follows:
$(a,b) + (c,d) = (a+c, b+d)$; $(a,b) \cdot (c,d) = (ac - bd, ad + bc)$.
Prove that C is a field. What famous field is C equivalent [isomorphic] to?

3) In each section, we define addition and multiplication on $\mathbb{R}^2$.
Decide, in each case, whether or not $F = (\mathbb{R}, +, \cdot)$ is a field

a. $(a,b) + (c,d) = (a+c, b+d)$
$(a,b) \cdot (c,d) = (ac, bd)$

b. $(a,b) + (c,d) = (a+c, b+d)$
$(a,b) \cdot (c,d) = (ac + 2bd, ad + bc)$

4) 1. Prove that, in a field, the 0 element is unique.
2. Prove that, in a field, the 1 element is unique.
3. Prove that, in a field, the additive inverse is unique.
4. Prove that, in a field, the multiplicative inverse is unique.

5) Let a, b be members of some field and let 0 be the additive identity.
Prove each of the following.

a. $a + a = a \Rightarrow a = 0$

b. $a \cdot 0 = 0 \cdot a = 0$

c. $a \cdot b = 0 \Leftrightarrow a = 0 \vee b = 0$

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6) Let a,b be members of some field and let 0 be the additive identity.
Prove each of the following.

a. $(-1)\cdot a=-a$

b. $(-a)\cdot b=a\cdot(-b)=-(a\cdot b)$

7) Prove the *cancellation law* of fields:
If $a\cdot c=b\cdot c$ and $c\neq 0$ then $a=b$. [a,b,c are elements of a given field.

8) [Solution of a linear equation in a field.]
Let $a\neq 0,b,c$ be elements of some field.
Prove that the equation (in *x*) $ax+b=c$ has a unique solution.

9) Let x,y be members of a field and suppose $x\cdot y\neq 0,1$.
Show that the expressions $(x - x\cdot y\cdot x)^{-1}$ and $x^{-1}+(y^{-1}-x)^{-1}$ are both well-defined, and that they are equal to one another.

10) Let F be a finite field of order n and let $0\neq a\in F$.
Prove that $a^k=1_F$ for some $k\in\mathbb{N}$. Recall: $a^k=\underbrace{a\cdot a\cdot\ldots\cdot a}_{k\text{ times}}$

11) Consider the field $\mathbb{Z}_7=\{\overline{0},\overline{1},\overline{2},\overline{3},\overline{4},\overline{5},\overline{6}\}$.

a. Define [informally] the field's addition $\oplus$ and multiplication $\odot$ operations.
What are the neutral elements (identities) for $\oplus$ and $\odot$?

b. Find the negatives (additive inverses) of $\overline{3}$ and $\overline{5}$.

c. Find the reciprocals (multiplicative inverses) of $\overline{4}$ and $\overline{5}$.

12) Let $\mathbb{Q}[\sqrt{2}]=\{x\in\mathbb{R}\mid x=a+b\sqrt{2},\ a\in\mathbb{Q},\ b\in\mathbb{Q}\}$.
Show that $\mathbb{Q}[\sqrt{2}]$ is a **subfield** of $\mathbb{R}$ with the same addition and multiplication.

13) a. Let $B=\mathbb{R}^{\mathbb{R}}=\{f:\mathbb{R}\to\mathbb{R}\}$ and define $+$ and $\cdot$ as pointwise addition, and ultiplication.That is, $(f+g)(x)=f(x)+g(x)$ and $(f\cdot g)(x)=f(x)\cdot g(x)$ for all $x\in\mathbb{R}$. Is $(B,+,\cdot)$ a field?

b. Let $A\subseteq B$ be the subset $\{f:\mathbb{R}\to\mathbb{R}\mid\forall x,\ f(x)\neq 0\}$.
Is A a subfield of B (with the inherited operations)?

14) Prove that the ring $\mathbb{Z}_n$ is a field if and only if n is prime.

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Answer Key

1) a. F doesn't satisfy all field axioms (existence of multiplicative inverses, distributive law).
Remark: F is not a field since it doesn't satisfy all the axioms.
b. F satisfies all field axioms.
Remark: $F = (\mathbb{R}, \oplus, \otimes)$ is a field!
c. F doesn't satisfy all field axioms (please watch the solution video for further explanation)
Remark: $F = (\mathbb{R}, \oplus, \otimes)$ is not a field!

2) F satisfies all field axioms. $C = (\mathbb{R}, \oplus, \otimes)$ is a field!
The "famous" field is the field of Complex Numbers, denoted $\mathbb{C}$.
The element (x, y) corresponds to $x + iy$.

3) a. This is not a field. b. This is not a field

4) This is a proof question, for full solution please watch our video.
5) This is a proof question, for full solution please watch our video.
6) This is a proof question, for full solution please watch our video.
7) This is a proof question, for full solution please watch our video.
8) This is a proof question, for full solution please watch our video.
9) This is a proof question, for full solution please watch our video.
10) This is a proof question, for full solution please watch our video.

11)

a. $\bar{a} \oplus \bar{b} = \overline{a+b}$
$\bar{a} \odot \bar{b} = \overline{ab}$

b. $-3 \equiv 4 (\bmod 7)$
$-5 \equiv 2 (\bmod 7)$

c. reciprocals of $\bar{4}$ is $\bar{2}$.
reciprocals of $\bar{5}$ is $\bar{3}$.

12) This is a proof question, for full solution please watch our video.

13) a. Not a field b. Not a subfield.
for full solution please watch our video

14) This is a proof question, for full solution please watch our video.

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