proprep

Workbook

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General Vector Spaces

Vector Spaces

Questions

Check if W is a subspace of $M_n[\mathbb{R}]$, where:

1) $W = \{A \mid A = A^T\}$.

2) W is the set of matrices which *commute* with a given matrix B.
That is, $W = \{A \mid AB = BA\}$.

3) W is the set of matrices whose determinant is 0. That is, $W = \{A \mid \det A = 0\}$.

4) W is the set of matrices which are equal to their own square.
That is, $W = \{A \mid A^2 = A\}$.

5) W is the set of upper-triangular matrices.

6) W is the set of matrices whose product with a given matrix B is 0.
That is, $W = \{A \mid AB = \mathbf{0}\}$.

7) W is the set of matrices whose trace is 0.
That is, $W = \{A \mid tr(A) = 0\}$.

8) W is the set of matrices such that the sum of each row is 0.

Check if W is a subspace of $P_n[\mathbb{R}]$, where:

9) W consists of the polynomials having 4 as a root. I.e., $W = \{p(x) \mid p(4) = 0\}$.

10) W consists of the polynomials with degree ≤ 4. I.e., $W = \{p(x) \mid \deg(p) \leq 4\}$.

11) W consists of the polynomials with integer coefficients.

12) W consists of the polynomials with only even powers of x in its terms.

13) W consists of the polynomials having degree n where $4 \le n \le 7$.

14) $W = \left\{ p(x) \middle| p(0) = 1 \right\}$.

Check if W is a subspace of $F[\mathbb{R}]$, where:

15) W consists of all even functions. I.e., $W = \left\{ f(x) \mid f(x) = f(-x) \text{ for all } x \in \mathbb{R} \right\}$.

16) W consists of all bounded functions. I.e.,
$W = \left\{ f(x) \mid \left| f(x) \right| \le M \text{ for all } x \in \mathbb{R}, \text{ for some } M > 0 \right\}$.

17) W consists of all continuous functions.

18) W consists of all differentiable functions.

19) W consists of all constant functions.

20) $W = \left\{ f(x) \mid \int_0^1 f(x)\,dx = 4 \text{ (assume } f \text{ is integrable)} \right\}$.

21) $W = \left\{ f(x) \mid f'(x) = 0 \text{ (assume } f \text{ is differentiable)} \right\}$.

22) $W = \left\{ f(x) \mid f'(x) = 1 \text{ (assume } f \text{ is differentiable)} \right\}$.

23) $W = \left\{ f \mid f(x+1) = f(x) \text{ for all } x \in \mathbb{R} \right\}$.

Check if W is a subspace of $\mathbb{C}^3[\mathbb{R}]$:

24) Check if W is a subspace of $\mathbb{C}^3[\mathbb{R}]$, where $W = \left\{ \langle z_1, z_2, z_3 \rangle \mid z_2 = \bar{z}_1, z_3 = z_1 + \bar{z}_1 \right\}$.

25) Check if $W = \left\{ \langle z_1, z_2, z_3 \rangle \mid z_2 = \bar{z}_1, z_3 = z_1 + \bar{z}_1 \right\}$ is a subspace of $\mathbb{C}^3$ (over the complex field $\mathbb{C}$).

Answer Key

1) Is a subspace
2) Is a subspace
3) Not a subspace
4) Not a subspace
5) Not a subspace
6) Not a subspace
7) Not a subspace
8) Not a subspace
9) Not a subspace
10) Not a subspace
11) Not a subspace
12) Is a subspace
13) Not a subspace
14) Not a subspace
15) Is a subspace
16) Is a subspace
17) Is a subspace
18) Is a subspace
19) Is a subspace
20) Not a subspace
21) Is a subspace
22) Not a subspace
23) Is a subspace
24) Is a subspace
25) Not a subspace

Linear Combination, Dependence and Span

Questions

1) We are given the following matrices in $M_2[\mathbb{R}]$:

$$A=\begin{bmatrix}4 & 1\\ 1 & 5\end{bmatrix},\ B=\begin{bmatrix}0 & 11\\ -5 & 3\end{bmatrix},\ C=\begin{bmatrix}2 & -5\\ 3 & 1\end{bmatrix},\ D=\begin{bmatrix}1 & 3\\ -1 & 2\end{bmatrix}$$

a. Are these matrices linearly dependent?
b. If so, try to write each of them as a linear combination of the rest.
c. Does A belong to $Sp\{B,C\}$?

2) We are given the following polynomials in $P_3[\mathbb{R}]$:

$$p_1(x)=4+x+x^2+5x^3\ ,\ p_2(x)=11x-5x^2+3x^3\ ,$$
$$p_3(x)=2-5x+3x^2+x^3\ ,\ p_4(x)=1+3x-x^2+2x^3$$

a. Are these polynomials linearly dependent?
b. If so, try to write each of them as a linear combination of the rest.
c. Does p_2 belong to $Sp\{p_1,p_3\}$?

3) We are given the following set of vectors in $\mathbb{R}^3$: $S=\{\langle c,2,4\rangle,\langle 2.4,a,2\rangle,\langle c,b,6\rangle,\langle b,2,a\rangle\}$

For which values of a,b,c is S linearly dependent?

4) We are given that the set $\{u,v,w\}$ of vectors is linearly independent in $V[F]$.

a. Is the set $\{u-v,u-w,u+v-2w\}$ linearly dependent?
b. If so, try to write each vector in the set as a linear combination of the others.

5) We are given that the set $\{u,v,w\}$ of vectors is linearly independent in $V[F]$.

a. Is the set $\{u+v,v+w,w\}$ linearly dependent?
b. If so, try to write vector in the set as a linear combination of the others.

6) We are given that the set $\{u,v,w\}$ of vectors is linearly independent in $V[F]$.

a. Is the set $\{u+2v+3w,4u+5v+6w,7u+8v+9w\}$ linearly dependent?
b. If so, try to write each vector in the set as a linear combination of the others.

7) Is the set of vectors $\{\langle 1,i,i-1\rangle,\langle i+1,i-1,-2\rangle\}$ linearly independent in $\mathbb{C}^3[\mathbb{C}]$?

8) Is the set of vectors $\{\langle 1,i,i-1\rangle,\langle i+1,i-1,-2\rangle\}$ linearly independent in $\mathbb{C}^3[\mathbb{R}]$?

Answer Key

1) a. Yes, they're linearly dependent.

b. $A = B + 2C$, $B = A - 2C$, $C = \frac{1}{2}A - \frac{1}{2}B$, $D = \frac{1}{4}A + \frac{1}{4}B$

c. Yes, follows from $A = B + 2C$.

2) a. Yes, they are linearly dependent.

b. $p_1 = p_2 + 2p_3$, $p_2 = p_1 - 2p_3$, $p_3 = \frac{1}{2}p_1 - \frac{1}{2}p_2$, $p_4 = \frac{1}{4}p_1 + \frac{1}{4}p_2$

c. Yes, follows from $p_2 = p_1 - 2p_3$.

3) For all values a,b,c S linearly is dependent.

4) a. Yes, they are linearly dependent.

b. $x = 2y - z$
$y = 0.5x + 0.5z$
$z = 2y - x$

5) a. No

b. N/A

6) a. Yes, they are linearly dependent.

b. $x = 2y - z$
$y = 0.5x + 0.5z$
$z = 2y - x$

7) No, the vectors are linearly dependent.

8) The vectors are linearly independent.

Vector Basis

Questions

1) Check if each of the following sets is a basis of $M_{2\times 2}[\mathbb{R}]$ (A.K.A $M_2[\mathbb{R}]$):

a. $\left\{\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix},\begin{bmatrix}5 & 6\\ 7 & 8\end{bmatrix},\begin{bmatrix}9 & 1\\ 2 & 3\end{bmatrix}\right\}$

b. $\left\{\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix},\begin{bmatrix}5 & 6\\ 7 & 8\end{bmatrix},\begin{bmatrix}9 & 1\\ 2 & 3\end{bmatrix},\begin{bmatrix}5 & 6\\ 7 & 2\end{bmatrix},\begin{bmatrix}5 & 16\\ 7 & 8\end{bmatrix}\right\}$

c. $\left\{\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix},\begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix},\begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix},\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right\}$

2) Check if each of the following sets is a basis of $P_2[\mathbb{R}]$ (deg ≤ 2 poly):

a. $\{1+x,\ x^2+2x+3\}$

b. $\{1+x,\ x^2+2x+3, 2x+4x^2, x-x^2\}$

c. $\{1+2x+3x^2,\ 4+5x+6x^2, 7+8x+10x^2\}$

Answer Key

1) a. No, the three vectors can't form a basis.

b. No, the five vectors can't form a basis.

c. Yes, the four vectors do form a basis.

2) a. No, the two vectors can't form a basis.

b. No, the four vectors can't form a basis.

c. Yes, the three vectors do form a basis.

Solution Space of Homogenous SLE

Questions

1) Let $U=\{A\in M_2[\mathbb{R}] \mid A=A^T\}$. Symmetric 2x2 matrices.

Find a basis and the dimension of U.

2) Let $U=\left\{A\in M_2[R] \;\middle|\; A\cdot\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\right\}$.

Find a basis and the dimension of U.

3) Let $U=\{p(x)\in P_3[\mathbb{R}] \mid p(1)=0\}$

Find a basis and the dimension of U.

Answer Key

1) $B_U=\left\{\begin{bmatrix}0&0\\0&1\end{bmatrix},\begin{bmatrix}0&1\\1&0\end{bmatrix},\begin{bmatrix}1&0\\0&0\end{bmatrix}\right\}$, $\dim U=3$.

2) $U=\left\{\begin{bmatrix}0&0\\0&0\end{bmatrix}\right\}$, $\dim U=0$, $B_U=\varnothing$ [empty set].

3) $B_U=\left\{p_1(x)=-1+x^3\ \ p_2(x)=-1+x^2, p_3(x)=-1+x\right\}$, $\dim U=3$

Subspaces

Questions

1) Consider the subspace of $M_2[\mathbb{R}]$ defined as follows:

$U = span\left\{\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} 4 & 1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & -1 \\ 1 & -3 \end{bmatrix}\right\}.$

Find a basis and the dimension of U .

2) Consider the subspace of $P_3[\mathbb{R}]$ defined as follows:

$U = span\{1 + x - x^2 + 2x^3, 4 + x - x^2 + x^3, 2 - x + x^2 - 3x^3\}$

Find a basis and the dimension of U .

3) Let $\mathbb{R}_3[x]$ be the space of polynomials in x of degree $\leq$ 3. Now consider a subset $U = \{p(x) = a_3x^3 + a_2x^2 + a_1x + a_0 : p(0) = 0, p(1) = 0\}$ and a subspace $V = Span(1, x^2) \subset \mathbb{R}_3[x]$.

Prove that $\mathbb{R}_3[x] = U \oplus V$.

4) Let W be a finite dimensional vector space. Let A and B be subspaces of W.

a. Prove that the set $U = \{a + b; a \in A, b \in B\}$ is a subspace of W.

b. Prove that it is the smallest subspace containing A and B.

c. Give example subspaces A and B in which $U = A \oplus B$.

Answer Key

1) $B_U = \left\{ \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & -3 \\ 3 & -7 \end{bmatrix} \right\}, \ \dim U = 2$

2) $B_U = \left\{ 1 + x - x^2 + 2x^3, -3x + 3x^2 - 7x^3 \right\}, \ \dim U = 2$

3) + 4) To view the answers to those exercises, please refer to the appropriate videos on site.

Change of Basis

Questions

1) Given the following two bases of $P_2[\mathbb{R}]$:

$B_1 = \{1+x, x, x+x^2\}$; and $B_2 = \{1+x^2, x+x^2, x^2\}$, and let $p(x) = a+bx+cx^2$,

be a general polynomial in $P_2[\mathbb{R}]$.

Compute $[p(x)]_{B_1}$, the coordinate vector of $p(x)$ relative to B_1 and B_2.

Find the change-of-basis matrix from B_1 to B_2.

2) Given the following two bases of $M_2[\mathbb{R}]$:

$$B = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$E = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Compute the coordinate vector of $X = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$, relative to B and E.

Find the change-of-basis matrix from B to E.

Answer Key

1) $[v]_{B_1} = \langle a, b-a-c, c\rangle$; $[v]_{B_2} = \langle a, b, c-a-b\rangle$, $[M]_{B_1}^{B_2} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$

2) $[X]_B = \langle x, \mathrm{y}-\mathrm{x}, z-y+x, t-z+y-x\rangle$

E is the elementary or standard basis of $M_2[\mathbb{R}]$: $[X]_E = \langle x, \mathrm{y}, z, t\rangle$

The change-of-basis matrix from B to E: $[M]_B^E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 1 \end{bmatrix}$