proprep

Workbook

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Linear Transformations

Linear Transformation Definition

Questions

Check if the following transformations are linear transformations:

1) $T:\mathbb{R}^2 \to \mathbb{R}^2$; $T[x,y]=[x+y,x-y]$

2) $T:\mathbb{R}^3 \to \mathbb{R}^3$; $T[x,y,z]=[x+y-2z,x+2y+z,2x+2y-3z]$

3) $T:\mathbb{R}^3 \to \mathbb{R}^2$; $T[x,y,z]=[2x+z,|y|]$

4) $T:\mathbb{R}^2 \to \mathbb{R}^3$; $T[x,y]=[xy,y,z]$

5) $T:\mathbb{R}^3 \to \mathbb{R}^3$; $T[x,y,z]=[x+1,x+y,x+z]$

6) $T:M_n[\mathbb{R}]\to M_n[\mathbb{R}]$; $T(A)=BA+AB$, where $B\in M_n[R]$ is some fixed matrix.

7) $T:M_n[\mathbb{R}]\to M_n[\mathbb{R}]$; $T(A)=A+A^T$

8) $T:M_n[\mathbb{R}]\to M_n[\mathbb{R}]$; $T(A)=|A|\cdot I$ ($|A|$ is the determinant of A)

9) $T:M_n[\mathbb{R}]\to M_n[\mathbb{R}]$; $T(A)=A\cdot A^T$

10) $T:M_n[\mathbb{R}]\to M_n[\mathbb{R}]$; $T(A)=A^3$

11) $T:P_3[\mathbb{R}]\to P_2[\mathbb{R}]$; $T(a+bx+cx^2+dx^3)=a+bx+cx^2$

12) $T:P_n[\mathbb{R}]\to P_n[\mathbb{R}]$; $T(p(x))=p(x+1)$

13) $T:P_n[\mathbb{R}]\to P_n[\mathbb{R}]$; $T(p(x))=p'(x)+p''(x)$

14) $T: P_n[\mathbb{R}] \to P_{2n}[\mathbb{R}]$; $T(p(x)) = p^2(x)$

15) Check if the following transformations are linear transformations:

a. $T: \mathbb{C}[\mathbb{R}] \to \mathbb{C}[\mathbb{R}]$; $T(z) = \overline{z}$

b. $T: \mathbb{C}[\mathbb{C}] \to \mathbb{C}[\mathbb{C}]$; $T(z) = \overline{z}$

16) For which value(s) of m is the following a linear transformation?

$T: \mathbb{R}^2 \to \mathbb{R}^2$; $T[x, y] = [m^2 x^{2m}, y^{2m} + x]$

17) Is there a linear transformation $T: \mathbb{R}^4 \to \mathbb{R}^3$, such that:

$T[1,2,-1,0] = [0,1,-1]$, $T[-1,0,1,1] = [1,0,0]$, $T[0,4,0,2] = [2,2,-2]$?

If there isn't, explain why.

If there is, find such a T and say whether this T is unique.

18) Is there a linear transformation $T: P_2[\mathbb{R}] \to P_2[\mathbb{R}]$, such that:

$T(1) = 4$, $T(4x + x^2) = x$, $T(1 - x) = x^2 + 1$?

If there isn't, explain why.

If there is, is it unique? If T is unique, find its formula.

19) Is there a linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$, such that:

$T[1,1,0] = [1,2,3]$, $T[0,1,1] = [4,5,6]$, $T[0,0,1] = [7,8,9]$?

If there isn't, explain why.

If there is, is it unique? If T is unique, find its formula.

20) Is there a linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$, such that:

$T[1,0,1] = [1,1,0]$, $T[0,1,1] = [1,2,1]$, $T[0,0,1] = [0,1,1]$?

If there isn't, explain why.

If there is, is it unique? If T is unique, find its formula.

Answer Key

1) T is linear.
2) T is linear.
3) T is not linear.
4) T is not linear.
5) T is not linear.
6) T is linear.
7) T is linear.
8) T is not linear.
9) T is not linear.
10) T is not linear.
11) T is linear.
12) T is linear.
13) T is linear.
14) T is not linear.
15) a. T is linear. b. T is not linear.
16) $m=2$
17) Yes, not unique $T[x,y,z,t]=\left[\frac{1}{2}y-x,\frac{1}{2}y,-\frac{1}{2}y\right]$.
18) Yes, unique T $\overline{T}[a,b,c]=[4a+3b-12c,c,4c-b]$.
19) Yes, unique T $T[x,y,z]=[4x-3y+7z,5x-3y+8z,6x-3y+9z]$.
20) Yes, unique T $T[x,y,z]=[x+y,y+z,z-x]$.

Image and Kernel

Questions

1) Let $T:\mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation defined by
$T[x,y,z,t]=[x+y,y-4z+t,4x+y+4z-t]$

a. Find a basis and the dimension of the kernel of T.
b. Find a basis and the dimension of the image of T.

2) Let $T:\mathbb{R}^3 \to \mathbb{R}^4$ be the linear transformation defined by
$T[x,y,z]=[x-4y-z,x+y,y-z,x+4z]$

a. Find a basis and the dimension of the kernel of T.
b. Find a basis and the dimension of the image of T.

3) Let $T:\mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation defined by

$$T[x,y,z,t]=\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 5 & -2 \\ 2 & 6 & 10 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}$$

a. Find a basis and the dimension of the kernel of T.
b. Find a basis and the dimension of the image of T.

4) Let $T:M_2[\mathbb{R}] \to M_2[\mathbb{R}]$ be the linear transformation defined by $T(A)=A\cdot\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}-\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\cdot A$.

a. Find a basis and the dimension of the kernel of T.
b. Find a basis and the dimension of the image of T.

5) Let $T:P_2[\mathbb{R}] \to P_2[\mathbb{R}]$ be the linear transformation defined by $T(p(x))=p(x+1)-p(x+4)$.

a. Find a basis and the dimension of the kernel of T.
b. Find a basis and the dimension of the image of T.

6) Let $D:P_3[\mathbb{R}] \to P_3[\mathbb{R}]$ be the linear transformation defined by $D(p(x))=p'(x)$.

a. Find a basis and the dimension of the kernel of D.
b. Find a basis and the dimension of the image of D.

7) Find a linear transformation $T:\mathbb{R}^3 \to \mathbb{R}^3$ whose image is spanned by $\{[4,1,4],[-1,4,1]\}$.

8) Find a linear transformation $T:\mathbb{R}^4 \to \mathbb{R}^3$ whose kernel is spanned by $\{[0,1,1,1],[1,2,3,4]\}$.

9) Let $T:V \to U$ be a linear transformation.
Prove that if $\dim(\operatorname{Im}T) = \dim(KerT)$, then the dimension of V is even.

10) Is it possible for a linear transformation $T:\mathbb{R}^4 \to \mathbb{R}^3$ to be one-to-one?

11) For the following linear transformation T, find $KerT$ and $\operatorname{Im}T$, and determine whether T is injective and/or surjective.

$T:Mat_{\mathbb{Z}}(2,3) \to Mat_{\mathbb{Z}}(3,2)$ given by $T\left(\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{pmatrix}\right) = \begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \\ A_{13} & A_{23} \end{pmatrix}$ for all $\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{pmatrix} \in Mat_{\mathbb{Z}}(2,3)$

Answer Key

1) a. $B_{\ker T}=\{[0,0,1,4]\}$, $\dim(KerT)=1$ b. $B_{\operatorname{Im}T}=\{[1,0,4],[0,1,-3],[0,0,1]\}$, $\dim(\operatorname{Im}T)=3$

2) a. $B_{KerT}=\{(0,0,0)\}$, $\dim(KerT)=0$ b. $B_{\operatorname{Im}T}=\{[1,1,0,1],[0,5,1,4],[0,0,-6,21]\}$, $\dim(\operatorname{Im}T)=3$

3) a. $B_{KerT}=\{[-7,3,0,1],[1,-2,1,0]\}$, $\dim(KerT)=2$

b. $B_{\operatorname{Im}T}=\{[1,1,2],[0,1,2]\}$, $\dim(\operatorname{Im}T)=2$

4) a. $B_{KerT}=\left\{\begin{bmatrix}-1 & 1\\ 0 & 0\end{bmatrix},\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right\}$, $\dim(KerT)=2$

b. $B_{\operatorname{Im}T}=\left\{\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix},\begin{bmatrix}1 & 0\\ 1 & -1\end{bmatrix}\right\}$, $\dim(\operatorname{Im}T)=2$

5) a. $B_{KerT}=\{1\}$, $\dim(KerT)=1$ b. $B_{KerT}=\{2x+5,1\}$, $\dim(\operatorname{Im}T)=2$

6) a. $B_{KerD}=\{1\}$, $\dim(KerD)=1$ b. $B_{\operatorname{Im}D}=\{x^2,x,1\}$, $\dim(\operatorname{Im}D)=3$

7) $T\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}4 & -1 & -1\\ 1 & 4 & 4\\ 4 & 1 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}$

8) $T[x,y,z,t]=[-x-y+z,-2x-y+t,0]$

9) Proved as shown in the video.

10) No, T is not one-to-one.

Isomorphism and Inverse

Questions

1) Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by $T[x,y,z]=[x-y+z,y+z,z-x]$.

True or false:

a. T is one-to-one.

b. T is onto.

c. T is an isomorphism.

d. T has an inverse. If it does, find it.

2) Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by $T[x,y,z]=[x-y+z,y+z,x+2z]$.

True or false:

a. T is one-to-one.

b. T is onto.

c. T is an isomorphism.

d. T has an inverse. If it does, find it.

3) Let $T:P_2[\mathbb{R}] \to \mathbb{R}^3$ be the linear transformation defined by $T(a+bx+cx^2)=[a+b+c,a-b,b-2c]$.

True or false:

a. T is one-to-one.

b. T is onto.

c. T is an isomorphism.

d. T has an inverse. If it does, find it.

4) Let $T:M_2[\mathbb{R}] \to P_3[\mathbb{R}]$ be the linear transformation defined by

$$T\begin{bmatrix} a & b \\ c & d \end{bmatrix}=a-b+(c+d)x+(a-c)x^2+dx^3.$$

True or false:

a. T is one-to-one.

b. T is onto.

c. T is an isomorphism.

d. T has an inverse. If it does, find it.

Answer Key

1) a. True b. True c. True

d. True, $T^{-1}\left[x,y,z\right]=\left[\frac{1}{3}\left(x+y-2z\right),\frac{1}{3}\left(2y-z-x\right),\frac{1}{3}\left(z+x+y\right)\right]$.

2) a. False b. False c. False d. False

3) a. True b. True c. True

d. True, $T^{-1}\left[a,b,c\right]=\left(0.4a+0.6b+0.2c\right)\mathbf{1}+(0.4a-0.4b+0.2c)\mathrm{x}+\left(0.2a-0.2b-0.4c\right)\mathrm{x}^2$

4) a. True b. True c. True

d. True, $T^{-1}\left[a,b,c,d\right]=\left[b+c-d,-a+b+c-d,b-d,d\right]$.

Composition of Linear Transformation

Questions

1) Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ and $S:\mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by $T[x,y,z]=[x,4x-y,x+4y-z]$, $S[x,y,z]=[x-z,y]$.
Find a formula, if possible, that defines $S+T$.

2) Let $S:\mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by $S[x,y,z]=[x-z,y]$.
Find a formula, if possible, that defines $4S$.

3) Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ and $S:\mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by $T[x,y,z]=[x,4x-y,x+4y-z]$, $S[x,y,z]=[x-z,y]$.
Find a formula, if possible, that defines $4S-10T$.

4) Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ and $S:\mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by $T[x,y,z]=[x,4x-y,x+4y-z]$, $S[x,y,z]=[x-z,y]$.
Find a formula, if possible, that defines TS , meaning function composition $T \circ S$.

5) Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ and $S:\mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by $T[x,y,z]=[x,4x-y,x+4y-z]$, $S[x,y,z]=[x-z,y]$.
Find a formula, if possible, that defines ST , meaning function composition $S \circ T$.

6) Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by $T[x,y,z]=[x,4x-y,x+4y-z]$.
Find a formula, if possible, that defines $T^2=TT$.

7) Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by $T[x,y,z]=[x,4x-y,x+4y-z]$.
Find a formula, if possible, that defines T^{-1}.

8) Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by $T[x,y,z]=[x,4x-y,x+4y-z]$.
Find a formula, if possible, that defines T^{-2}.

9) Let $S:\mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by $S[x,y,z]=[x-z,y]$.
Find a formula, if possible, that defines $S^2=SS$.

Answer Key

1) $S+T$ can't be defined.

2) $\left[4(x-z),4y\right]$

3) $4S-10T$ can't be defined.

4) $T\circ S$ can't be defined.

5) $ST\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-4y+z\\4x-y\end{bmatrix}$

6) $\left[x,y,16x-8y+z\right]$

7) $T^{-1}\left[x,y,z\right]=\left[x,4x-y,17x-4y-z\right]$

8) $T^{-2}\left[x,y,z\right]=\left[x,y,-16x+8y+z\right]$

9) $S^2=SS$ can't be defined.