proprep

Workbook

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First Order Linear Equations

First Order Separable Equations

Questions

Solve the following equations:

1) $\frac{dy}{dx}=\frac{x^2}{y}$, $(y\neq 0)$

2) $(1-x)y'=y^2$

3) $yy'\sqrt{1+x^2}+x\sqrt{1+y^2}=0$

4) $(x-1)\frac{dy}{dx}=4y$; $y(2)=1$

5) $\frac{dy}{dx}=xy+3y-3x-9$; $y(1)=-1$

6) $\left(x^2y-2+2x^2-y\right)dx-\left(xy^2-4-4x+y^2\right)dy=0$

7) $dy=2t\left(y^2+4\right)dt$

8) $\frac{dx}{dt}=x^2-2x+2$

9) $y'+y^2\sin x=0$; $y(\pi)=1$

10) $\frac{dy}{dx}=y\sec^2 x$; $y(0)=5$

11) $\frac{dy}{dx}=\frac{xy^3}{\sqrt{1+x^2}}$; $y(0)=1$

12) Let $y(t)$ denote the amount of substance (or population) that is either growing or decaying. Assume that the time rate of change of this amount of substance is proportional to the amount of substance present (i.e. the amount $y(t)$ grows/decays exponentially). Assume that at the start time $t=0$ the amount is y_0, and find a formula for the amount at any given time t.

13) If the population of the earth was found to be 4 billion in 1980, and is increasing at a rate of 2% per year.

a. Find the population of the earth in 2010.
b. Find the population of the earth in 1974.
c. When will a population of 50 billion be reached?

Answer Key

1) $y=\pm\sqrt{\frac{2}{3}x^3+k}$

2) $y=\frac{1}{\ln|1-x|-c}$; $y=0$

3) $\sqrt{1+y^2}=-\sqrt{1+x^2}+c$

4) $\frac{1}{4}\ln|y|=\ln|x-1|$

5) $\ln|y-3|=\frac{x^2}{2}+3x+\ln 4-3.5$

6) $y=-2$

7) $y=2\tan\left(2t^2+k\right)$

8) $x=1+\tan\left(t+c\right)$

9) $y=-\frac{1}{\cos x}$

10) $\ln|y|=\tan x+\ln 5$

11) $\frac{1}{-2y^2}=\sqrt{1+x^2}-1.5$

12) $y\left(t\right)=y_0e^{kt}$

13) a. 7.28 billion b. 4.51 billion c. 126 years, year 2016.

Homogeneous Equations

Questions

Solve the following equations:

1) $\left(y^3+x^3\right)dx+xy^2dy=0$

2) $y'=\dfrac{4y-3x}{2x-y}$

3) $y^2+x^2y'=xyy'$

4) $\left(3xy+y^2\right)dx+\left(x^2+xy\right)dy=0$

5) $\left(x-y\cos\dfrac{y}{x}\right)dx+x\cos\dfrac{y}{x}dy=0$

6) $y'=\dfrac{2xye^{(x/y)^2}}{y^2+y^2e^{(x/y)^2}+2x^2e^{(x/y)^2}}$

7) $\left(y+\sqrt{x^2+y^2}\right)dx-xdy=0\ ;\ y(1)=0$

8) $\left(2x^2t-2x^3\right)dt+\left(4x^3-6x^2t+2xt^2\right)dx=0$

9) Answer the following Questions

a. Find n so that the o.d.e $\left(y^2+x^2\right)dx+xy^ndy=0$ will be homogeneous.

b. Solve the above o.d.e for the value of n you found in part a.

Answer Key

1) $y=x\cdot v=-\dfrac{x}{2^{1/3}}$

2) $y=-3x$

3) $y=v\cdot x=0$

4) $y=-2x$

5) $\ln|x|=-\sin\left(y/x\right)+c$

6) $y=0$

7) $\ln x=\sinh^{-1}\left(\dfrac{y}{x}\right)$

8) $x(t)=t$

9) a. $n=1$ b. $\ln|x|=-\dfrac{1}{4}\ln\left(1+2\left(\dfrac{y}{x}\right)^2\right)+c$

Homogeneous after Substitution

Questions

Solve the following equations:

1) $\frac{dy}{dx} = \frac{x+y+1}{x+y+2}$

2) $(x+2y+3)dx + (2x+4y-1)dy = 0$

3) $\frac{dy}{dx} = \frac{2y-x+5}{2x-y-4}$

4) $\frac{dx}{dy} = \frac{3+x+2y}{1+x+y}$

5) $(2x+y-3)dx + (x+y-1)dy = 0$

Answer Key

1) $x = \frac{1}{2}z + \frac{1}{4}\ln(2z+1) + \frac{1}{4} + c \;\; ; \;\; y = -x - 1.5$

2) $0 = 14y - (x+2y+3)^2 + k$

3) $y = -x - 1 \;,\; y = x - 3$

4) $y = -\sqrt{0.5}x - 2 + \sqrt{0.5}$

5) $\ln|x-2| = \frac{1}{2}\ln\left(2 + 2\frac{y+1}{x-2} + \left(\frac{y+1}{x-2}\right)^2\right) + c$

Exact Equations

Questions

Solve the following equations:

1) $\left(2x^3+3y\right)dx+(3x+y-1)dy=0$

2) $\left(y^2-\frac{y}{x\left(x+y\right)}+2\right)dx+\left(\frac{1}{x+y}+2y\left(x+1\right)\right)dy=0$

3) $\left(y^2e^{xy^2}+4x^3\right)dx+\left(2xye^{xy^2}-3y^2\right)dy=0$

4) $\left(y\cos x+2xe^y\right)dx+\left(\sin x+x^2e^y-1\right)dy=0$

5) $\left(1+y^2\sin 2x\right)dx-2y\cos^2 xdy=0$

6) $\left(2x^2t-2x^3\right)dt+\left(4x^3-6x^2t+2xt^2\right)dx=0$

7) Given the differential equation $\left(3x^2+ye^{xy}\right)dx+\left(2y^3+kxe^{xy}\right)dy=0$.

a. Find the constant k which make the equation exact.

b. Solve the equation for the value of k found in part a.

Answer Key

1) $0.5x^4+3yx+0.5y^2-y=c$

2) $y^2x-\ln|x|+\ln|x+y|+y^2=c$

3) $e^{xy^2}+x^4-y^3=c$

4) $y\sin x+x^2e^y-y=c$

5) $x-\frac{y^2\cos 2x}{2}-\frac{y^2}{2}=c$

6) $x^2t^2-2x^3t+x^4=c$

7) a. $k=1$ b. $x^3+e^{xy}+\frac{y^4}{2}=c$

Integration Factor

Questions

Solve the following equations:

1) Show that the differential equation $x^2y^3 + x\left(1+y^2\right)y' = 0$ is not exact, and solve it using $\frac{1}{xy^3}$ as an integration factor.

2) Show that the equation $\left(\frac{\sin y}{y} - 2e^{-x}\sin x\right)dx + \left(\frac{\cos y + 2e^{-x}\cos x}{y}\right)dy = 0$ is not exact, and solve it using ye^x as an integration factor.

3) Show that the differential equation $(x+2)\sin y\,dx + x\cos y\,dy = 0$ is not exact, and solve it using xe^x as an integration factor.

4) $\left(x^2 + y^2 + x\right)dx + (xy)dy = 0$

5) $\left(x - x^2 - y^2\right)dx + ydy = 0$

6) $\left(2xy^3 + y^4\right)dx + \left(xy^3 - 2\right)dy = 0$

7) $\left(y^2 - y\right)dx + xdy = 0$

8) $\left(y - xy^2\right)dx + \left(x + x^2y^2\right)dy = 0$

9) $y' = \frac{3yx^2}{x^3 + 2y^4}$, $y(1) = -1$

10) Given the non-exact ODE: $M(x,y)dx + N(x,y)dy = 0$.

a. Prove that if $\frac{M_y - N_x}{N} = f(x)$, then $e^{\int f(x)dx}$ is an integration factor.

b. Prove that if $\frac{M_y - N_x}{M} = g(y)$, then $e^{-\int g(y)dy}$ is an integration factor.

11) Given the ODE $\left(y^4 - 4xy\right)dx + \left(2xy^3 - 3x^2\right)dy = 0$, find an integration factor for the equation which is a function of x, y alone. I.e. find an I.F. of the form $\mu(x,y)$.

12) Given the ODE $\left(y^4-4xy\right)dx+\left(2xy^3-3x^2\right)dy=0$,

find an integration factor for it of the form $\mu(x+y)$.

13) Given the ODE $\left(x^2y^3\right)dx+\left(x+xy^2\right)dy=0$, find an integration factor for the equation, which is a function of $x^\alpha y^\beta$ alone. I.e. find an I.F. of the form $\mu\left(x^\alpha y^\beta\right)$.

14) Given the ODE: $M(x,y)dx+N(x,y)dy=0$.

a. Find a condition on the equation in order for it to have an integration factor for the equation which is a function of x,y alone.

b. Use part a. to find an integration factor for the equation: $\left(y-xy^2\ln x\right)dx+xdy=0$.

15) Given the ODE: $M(x,y)dx+N(x,y)dy=0$, find a condition on the equation in order for it to have an integration factor for the equation which is a function of $x+y$ alone.

16) Given the ODE: $M(x,y)dx+N(x,y)dy=0$, find a condition on the equation in order for it to have an integration factor for the equation which is a function of $\frac{x}{y}$ alone.

Answer Key

1) $0.5x^2 + \frac{y^{-2}}{-2} + \ln|y| = c$

2) $e^x \sin y + 2y\cos x = c$

3) $\sin y \cdot e^x \cdot x^2 = c$

4) $0.25x^4 + 0.5x^2y^2 + \frac{x^3}{3} = c$

5) $\frac{1}{2}\ln\left(x^2 + y^2\right) - x = c$

6) $x^2 + xy + \frac{1}{y^2} = c$

7) $x - \frac{x}{y} = c$

8) $-\ln x - \frac{1}{xy} + y = c$

9) $-\frac{x^3}{y} + \frac{2y^3}{3} = \frac{1}{3}$

10) Solution in the recording.

11) $\mu(xy) = (xy)^2$

12) $\mu(x+y) = (x+y)^2$

13) $\mu = \frac{1}{xy^3}$

14) a. $\mu = e^{\int \frac{M_y - N_x}{y \cdot N - x \cdot M}}$

b. $\left(\frac{1}{x^2 y} - \frac{1}{x}\ln x\right)dx + \frac{1}{xy^2}dy = 0$

15) $\mu = e^{\int \frac{M_y - N_x}{N - M}}$

16) $\mu = e^{\int \frac{y^2\left(M_y - N_x\right)}{y \cdot N + x \cdot M}}$

Linear Equations

Questions

Solve the following equations:

1) $y'+2xy=4x$

2) $xy'=y+x^3+3x^2-2x\ ,\ (x\neq 0)$

3) $(x-2)y'=y+2(x-2)^3\ \ (x>2)$

4) $x^3y'+(2-3x^2)y=x^3\ ,\ (x>0)$

5) $\frac{dy}{dt}+y=2+2t\ ;\ y(0)=1$

6) $\frac{dy}{dx}+y\cot x=5e^{\cos x}\ ,\ (\sin x>0)$

7) $y'-2y\cot x=1\ ,\ (\sin x>0)$

8) $x^2z'+2xz=\cos x\ ;\ z(\pi)=0$

Answer Key

1) $y=2+C\cdot e^{-x^2}$

2) $y=x\left[\frac{x^2}{2}+3x-2\ln x+C\right]\ ,\ (x>0)$

3) $y=(x-2)\left[x^2-4x+C\right]$

4) $y=\frac{1}{2}x^3+C\cdot x^3e^{\frac{1}{x^2}}$

5) $y=2t+e^{-t}$

6) $y=\frac{1}{\sin x}\left[-5e^{\cos x}+C\right]$

7) $y=\sin^2 x\left[-\cot x+C\right]$

8) $z=\frac{\sin x}{x^2}$

Bernoulli Equations

Questions

Solve the following equations:

1) $x^2 y' + 2xy - y^3 = 0$

2) $\left(x^2+1\right)y' - 2xy - y^2 = 0$

3) $x\dfrac{dy}{dx} - 2y = x^2 y^{1/2}$

4) $y' - \left(\dfrac{1}{x} + 5x^4\right)y = -x^3 y^2 \;\;,\;\; y(1) = 2.5$

5) $z' - \cot x \cdot z = \dfrac{1}{\sin x} z^3 \;,\; \left(0 < x < \pi\right)$

Answer Key

1) $y = \pm \dfrac{1}{\sqrt{\dfrac{2}{5x} + C \cdot x^4}}$

2) $y = \dfrac{x^2+1}{-x+C}$

3) $y = x^2\left(\dfrac{x}{2} + C\right)^2$

4) $y = \dfrac{5xe^{x^5}}{e^{x^5} + e}$

5) $z = \pm\sqrt{\dfrac{\sin^2 x}{2\cos x + C}}$

Ricatti Equations

Questions

Solve the following equations:

1) $y' = e^{2x} + \left(1 + \frac{5}{2}e^{x}\right)y + y^2$

2) $y' = -\left(1 + x + x^2\right) - \left(2x + 1\right)y - y^2$

3) $y' = 1 + \left(x - y\right)^2$

4) $y' = 1 + x + 2x^2\cos x - \left(1 + 4x\cos x\right)y + 2y^2\cos x$

Answer Key

1) $y\left(x\right) = -0.5e^{x} + \frac{e^{x}}{-\frac{2}{3} + Ce^{-1.5x}}$

2) $y\left(x\right) = -x + \frac{1}{1 + Ce^{x}}$

3) $y\left(x\right) = x + \frac{1}{-x + C}$

4) $y\left(x\right) = x + \frac{1}{\cos x - \sin x + Ce^{x}}$

Existence and Uniqueness

Questions

1) Given the problem: $y' = -\frac{1}{2}x + \sqrt{\frac{1}{4}x^2 + y}$, $y(2) = -1$.

a. Prove that $y_1(x) = -x + 1$, $y_2(x) = -\frac{1}{4}x^2$ are both solutions.

b. Explain why this non-uniqueness doesn't contradict the Uniqueness Theorem.

2) Given the problem: $y(0) = 0$, $y' = \sqrt[3]{y} + 4$.

a. Prove that the problem satisfies the conditions of the Existence Theorem.

b. Prove that the problem does not satisfy the conditions of the Uniqueness Theorem.

c. Prove that the problem does have a unique solution and find it.

3) Solve the problem: $y' = (x^2 + y^2)\cos\left(\frac{\pi}{2} - y\right) + x^2 \sin y$, $y(4) = 0$.

4) Given the problem: $y' = (y-1)(x^2 + y)^5$, $y(0) = 4$.

a. Show that any solution of the problem is bounded from below.

b. Show that any solution of the problem is increasing in all its domain.

Answer Key

1) To view the answer to this exercise, please refer to the appropriate video on site.

2) To view the answer to this exercise, please refer to the appropriate video on site.

3) $y(x) = 0$

4) To view the answer to this exercise, please refer to the appropriate video on site.

Clairaut Equation

Questions

1) Given the differential equation: $y = x\frac{dy}{dx} - \cos\left(\frac{dy}{dx}\right)$.

Derive the **family of solutions** and the **singular solution**.

Answer Key

1) Family of solutions: $y_1(x) = Ax + B$,

Single Solution: $y_2(x) = -\left[\arcsin(x) + \sqrt{1 - x^2}\right] + C$.