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Workbook

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Rolle`s Theorem and The Mean Value Theorem

The Mean Value Theorem

Questions

1) Prove that $\frac{b-a}{b}<\ln\left(\frac{b}{a}\right)<\frac{b-a}{a} \qquad (0<a<b)$.

2) Prove that $\frac{b-a}{\cos^2 a}<\tan b-\tan a<\frac{b-a}{\cos^2 b} \qquad \left(0<a<b<\frac{\pi}{2}\right)$.

3) Prove that $(a-b)e^{-a}<e^{-b}-e^{-a}< \quad (a-b)e^{-b} \qquad (a<b)$.

4) Prove that $\frac{b-a}{1+b^2}<\arctan b-\arctan a<\frac{b-1}{1+a^2} \qquad (0>a>b)$.

5) Prove that $\frac{b-a}{\sqrt{1-a^2}}<\arcsin b-\arcsin a<\frac{b-a}{\sqrt{1-b^2}} \qquad (0<a<b<1)$.

6) Prove that $\frac{b-a}{\sqrt{1+b^2}}<\frac{\operatorname{arcsinh}(b)-\operatorname{arcsinh}(a)}{b-a}<\frac{b-a}{\sqrt{1+a^2}} \qquad (0<a<b)$.

7) Prove that $\frac{b-a}{1-a^2}<\operatorname{arctanh}(b)-\operatorname{arctanh}(a)<\frac{b-a}{1-b^2} \qquad (0<a<b<1)$.

8) Prove that $\sqrt[n]{b}\cdot\frac{b-a}{n\cdot b}<\sqrt[n]{b}-\sqrt[n]{a}<\sqrt[n]{a}\cdot\frac{b-a}{n\cdot a} \qquad (0<a<b)$.

9) Prove that $\frac{2b(b-a)}{b^2+1}<\ln\left(\frac{b^2+1}{a^2+1}\right)<\frac{2a(b-a)}{a^2+1} \qquad (0<a<b)$.

10) Prove that $x<\tan x<\frac{x}{\cos^2 x} \qquad \left(0<x<\frac{\pi}{2}\right)$.

11) Prove that $\frac{x}{1+x^2}<\arctan x<x \qquad (x>0)$.

12) Prove that $\frac{x}{\sqrt{1+x^2}}<\operatorname{arcsinh}(x)<x \qquad (x>0)$.

13) Prove that $x<\operatorname{arctanh}(x)<\frac{x}{1-x^2} \qquad (0<x<1)$.

14) Prove that $1+x<e^x<1+xe^x \qquad (x>0)$.

15) Prove that $\sin x \le x \quad (x > 0)$.

16) Prove that $\tan x < 4x \quad \left(0 < x < \frac{\pi}{3}\right)$.

17) Prove that $\arctan x > \ln(1+x) \quad (0 < x < 1)$.

18) Prove that $|\sin x_2 - \sin x_1| \le |x_2 - x_1|$.

19) Prove that $|\cos x_2 - \cos x_1| \le |x_2 - x_1|$.

20) Prove that $|\arctan y - \arctan x| < |y - x|$.

21) Prove that $|\tan y - \tan x| \le 8|\sin y - \sin x|, \quad x, y \in \left[0, \frac{\pi}{3}\right]$.

22) Prove that $\frac{1}{3} < \ln\left(\frac{3}{2}\right) < \frac{1}{2}$.

23) Prove that $\frac{1}{2\sqrt{2}} + 1 < \sqrt{2} < 1.5$.

24) Prove that $\frac{3}{25} + \frac{\pi}{4} < \arctan\left(\frac{4}{3}\right) < \frac{1}{6} + \frac{\pi}{4}$.

25) Prove that $\frac{\sqrt{3}}{15} + \frac{\pi}{6} < \arcsin(0.6) < \frac{1}{8} + \frac{\pi}{6}$.

26) Let $f(x)$ be a function which is differentiable for all x and satisfies $|f'(x)| \le 5$.

It is known that $f(1) = 3$, $f(4) = 18$. Prove that $f(2) = 8$.

27) Let $f(x)$ be a function which is differentiable for all x and satisfies $|f'(x)| \le 7$.

It is known that $f(1) = 3$, $f(4) = 18$. Prove that $4 \le f(2) \le 10$.

*Proof questions- for the solution see the videos.

Rolle's Theorem

Example Questions

1) Check if the given function f satisfies the conditions of Rolle's Theorem on the given interval. If so, find all the values of c as in the conclusion of Rolle's Theorem.

a. $f(x) = x^3 - 3x^2 + 2x$ on $[0,2]$.

b. $f(x) = \dfrac{x^2 - 1}{x - 2}$ on $[-1,1]$.

2) Given: $f(x) = \dfrac{1}{(x-3)^2}$. Show that $f(1) = f(5)$, but that there is no value c in $1 < c < 5$ such that $f'(c) = 0$. Does this contradict Rolle's Theorem ?

3) Prove that the equation $x^2 + x^3 + 5x = 1$ has exactly one solution.
Hint: show at least one solution and at most one solution.

Answer Key

1) a. $f(0)=0,\ \ f(2)=0$; Yes, it is continuous and differentiable

$$c=\frac{3\pm\sqrt{3}}{3}\cong 1.6\ and\ -0.4$$

b. $f(-1)=0,\ \ f(1)=0$; Yes, it is continuous and differentiable

$$c=2-\sqrt{3}$$

2-3) Refer to the video.

Advanced Exercises

Questions

1) Does there exist a differentiable function $f:[0,2]\to\mathbb{R}$ satisfying $f(0)=-1,\ f(2)=4$, and $f'(x)\le 2$ for all $x\in[0,2]$?

2) Let $f:[0,1]\to\mathbb{R}$ be differentiable such that $|f'(x)|<1$ for all $x\in[0,1]$.
Show that there exists at most one $c\in[0,1]$ such that $f(c)=c$.

3) Let $f:\mathbb{R}\to\mathbb{R}$ be differentiable such that for some $\alpha\in\mathbb{R}$, $|f'(x)|\le\alpha<1$ for all $x\in\mathbb{R}$.
Let $a_1\in\mathbb{R}$ and define a sequence (a_n) recursively by $a_{n+1}=f(a_n)$. Show that (a_n) converges.

4) Let $f:[0,1]\to\mathbb{R}$ be twice differentiable. Suppose that the line segment joining the points $(0,f(0))$ and $(1,f(1))$ intersect the graph of f at a point $(a,f(a))$, where $0<a<1$.
Show that there exists $x_0\in[0,1]$ such that $f''(x_0)=0$.

5) Let $f:[0,1]\to\mathbb{R}$ be continuous. Suppose f is differentiable on $(0,1)$, and $\lim\limits_{x\to 0^+} f'(x)=L$ for some $L\in\mathbb{R}$. Show that there exists $f_+'(0)=L$.

6) Let $f:[0,1]\to\mathbb{R}$ be differentiable and $f(0)=0$.
Suppose that $|f'(x)|<|f(x)|$ for all $x\in[0,1]$. Show that $f(x)=0$ for all $x\in[0,1]$.

7) Let $f:[0,\infty)\to\mathbb{R}$ be continuous and $f(0)=0$. Suppose that $f'(x)$ exists for all $x\in(0,\infty)$, and f' is increasing on $(0,\infty)$. Show that the function $g(x)=\frac{f(x)}{x}$ is increasing on $(0,\infty)$.

8) Let $a\ge 0$ and $f:[a,b]\to\mathbb{R}$ be differentiable. Using Cauchy's mean value theorem, show that there exist $c_1,c_2\in(a,b)$ such that $\dfrac{f'(c_1)}{a+b}=\dfrac{f'(c_2)}{2c_2}$.

9) Let $f:[a,b]\to\mathbb{R}$ be differentiable. If $f'(x)\ne 0$ for all $x\in[a,b]$,
show that either $f'(x)\ge 0\ \forall x\in[a,b]$ or $f'(x)\le 0\ \forall x\in[a,b]$.

10) Let $f:[a,b]\to\mathbb{R}$ be differentiable and let $\alpha\in\mathbb{R}$ be such that $f'(a)<\alpha<f'(b)$. Define $g(x)=f(x)-\alpha x$ for all $x\in[a,b]$.

a. Show that there exists $c\in[a,b]$ such that $g'(c)=0$.
Hint: prove by contradiction, noting that $g'(a)<0$ and $g'(b)<0$.

b. From the above, conclude that if a function f is differentiable on an interval $[a,b]$, then f' has the Intermediate Value Property on $[a,b]$.

11) Let f be differentiable on $[a,b]$ $(a<b)$. Show that there exist $c_1,c_2,c_3\in(a,b)$ such that $c_2\neq c_3$ and $f'(c_2)+f'(c_3)=2f'(c_1)$.

12) Suppose $f:[0,1]\to\mathbb{R}$ is continuous and $\int_0^1 f(t)dt=1$.

a. Show that there exists $c\in(0,1)$ such that $f(c)=1$.

b. Show that there exist $c_1\neq c_2$ in $(0,1)$ such that $f(c_1)+f(c_2)=2$.

13) Let $f:[0,1]\to\mathbb{R}$ be such that $|f'(x)|<10$ for all $x\in(0,1)$ and let (x_n) be a sequence in $(0,1)$ satisfying the Cauchy criterion. Show that the sequence $(f(x_n))$ converges.

14) Let $f:[0,1]\to[0,1]$ be such that $f'(x)<0$ for all $x\in[0,1]$.
Show that there is one and only one $c\in[0,1]$ such that $f(c)=c^2$.

15) Let $f:[0,1]\to\mathbb{R}$ and $a_n=f\left(\frac{1}{n}\right)-f\left(\frac{1}{n+1}\right)$, $n=1,2,\ldots$ Show that:

a. if f is continuous, then $\sum_{n=1}^{\infty}a_n$ converges;

b. if f is differentiable and $|f'(x)|<\frac{1}{2}$ $\forall x\in[0,1]$, then $\sum_{n=1}^{\infty}a_n(\cos n)\sqrt{n}$ converges.

*For the solutions go watch the videos- Proof questions.

Darboux's Mean Value Theorem

Questions

1) Does there exist a differentiable function f such that $f'(x)=\begin{cases}4x & ; x<1\\ x-1 & ; x\ge 1\end{cases}$.

2) Does there exist a differentiable function f such that $f'(x)=\begin{cases}4 & ; x=0\\ x^2 & ; x\ne 0\end{cases}$.

3) Does there exist a differentiable function f such that $f'(x)=\begin{cases}1 & ; x=0\\ \dfrac{1}{x^2} & ; x\ne 0\end{cases}$.

4) a. Let $f:[a,b]\to\mathbb{R}$ be differentiable and let $x_0\in(a,b)$.
Show that x_0 can not be a removable discontinuity of f'.

b. Does there exist a differentiable function f such that $f'(x)=\begin{cases}4 & ; x=0\\ x^2 & ; x\ne 0\end{cases}$?

5) a. Let $f:[a,b]\to\mathbb{R}$ be differentiable and let $x_0\in(a,b)$.
Show that x_0 can not be a type I jump discontinuity of f'.

b. Does there exist a differentiable function f such that $f'(x)=\begin{cases}x+1 & ; x\ge 1\\ 4x & ; x<1\end{cases}$?

6) a. Let $f:[a,b]\to\mathbb{R}$ be differentiable and let $x_0\in(a,b)$.
Show that x_0 can not be a type II infinite discontinuity of f'. I.e. $f'_-(x_0)\ne\pm\infty$, $f'_+(x_0)\ne\pm\infty$.

b. Does there exist a differentiable function f such that $f'(x)=\begin{cases}0 & ; x=0\\ \dfrac{1}{x^2} & ; x\ne 0\end{cases}$?

7) a. Let $f:[0,1]\to\mathbb{R}$ be continuous. Suppose f is differentiable on $(0,1)$, and $\lim\limits_{x\to 0^+} f'(x)=L$ for some $L\in\mathbb{R}$. Show that there exists $f_+'(0)=L$.

b. Let $f:[0,1]\to\mathbb{R}$ be continuous. Suppose f is differentiable on $(0,1)$, and $\lim\limits_{x\to 0^+} f'(x)=\infty$. Show that there exists $f_+'(0)=\infty$. Similarly for $-\infty$.

8) Consider the Dirichlet function $D(x)=\begin{cases}1; & x\in\mathbb{Q}\\ 0; & x\notin\mathbb{Q}\end{cases}$, restricted to $(0,1)$.

Is it possible that $D(x)=f'(x)$ where $f:[0,1]\to\mathbb{R}$ is differentiable?

*For the solutions go watch the videos- Proof questions.