proprep™

Workbook

The Argument Principle

The Argument Principle

Questions

1) Compute $\int\limits_{|z|=2} \frac{2z}{z^2+1} dz$ using the Argument Principle.

Reminder: $\oint\limits_{\gamma} \frac{f'(z)}{f(z)} dz = 2\pi i(N-P)$

2) Let $f(z) = \frac{(z+3)(z+1)}{z^2}$.

Compute $\int\limits_{|z|=2} \frac{f'(z)}{f(z)} dz$ using the Argument Principle.

Reminder: $\oint\limits_{\gamma} \frac{f'(z)}{f(z)} dz = 2\pi i(N-P)$

3) Compute $\int\limits_{|z|=1} \frac{3\sin 2z}{\sin^2 z - 0.5} dz$ using the Argument Principle.

Reminder: $\oint\limits_{\gamma} \frac{f'(z)}{f(z)} dz = 2\pi i(N-P)$

4) Let $f(z) = \frac{(z-2)^2}{(e^{2z}-1)^2 z^3}$.

(a) Find the zeros and poles of f (and their orders) in the domain $|z| < 4$.

(b) Compute $\frac{1}{2\pi i} \int\limits_{|z|=4} \frac{f'(z)}{f(z)} dz$ using the Argument Principle.

Reminder: $\frac{1}{2\pi i} \oint\limits_{\gamma} \frac{f'(z)}{f(z)} dz = N-P$.

5) Let $f(z)=\dfrac{\cos z}{\sin z(1-\cos z)^2}$.

(a) Find the zeros and poles of f (and their orders) in $\mathbb{C}$.

(b) Compute $\dfrac{1}{2\pi i}\displaystyle\int_{|z|=4}\dfrac{f'(z)}{f(z)}dz$ using the Argument Principle.

Reminder: $\dfrac{1}{2\pi i}\displaystyle\oint_{\gamma}\dfrac{f'(z)}{f(z)}dz=N-P$

6) Define the closed contour $\gamma:[0,2\pi]\to\mathbb{C}$ by

$$\gamma(t)=\frac{\left[\cos 2t-\cos t+i(\sin 2t-\sin t)+\frac{1}{4}\right]e^{\frac{\cos t-2-i\sin t}{5-4\cos t}}}{\sin\left(\cos t+i\sin t\right)-i\sin t-\cos t}.$$

Find the winding number $n(\gamma;0)$ of γ around 0.

Formula: $n(\gamma;a)=\dfrac{1}{2\pi i}\displaystyle\oint_{\gamma}\dfrac{1}{z-a}dz$.

The Argument Principle

7) Use the Argument Principle to compute the number of zeros N for the polynomial
$p(z)=z^2+1$ in $\operatorname{Im} z>0$ (the upper half-plane).

8) Use the Argument Principle to compute the number of zeros N for the polynomial
$p(z)=z^4+8z^3+3z^2+8z+3$ in $\operatorname{Re} z>0$ (the right half-plane).

Rouché's Theorem

9) Find the number of zeros of $h(z)=z^4+5z+1$ in
$D=\{z\in\mathbb{C}:|z|<1\}=D(0,1)$.

10) Find the number of zeros of $h(z)=5z^4+z+1$ in
$D=\{z\in\mathbb{C}:|z|<1\}=D(0,1)$.

11) Find the number of zeros of $h(z)=z^4+5z+4$ in $D=\{z\in\mathbb{C}:|z|<2\}=D(0,2)$.

12) Find the number of zeros of $h(z)=z^5+3z+1$ in $D=\{z\in\mathbb{C}:1<|z|<2\}$.

13) Find the number of zeros of $h(z)=z^4-10z+1$ in $D=\{z\in\mathbb{C}:1<|z|<3\}$.

14) Find the number of zeros of $h(z)=\frac{1}{2}z^6-5z^4+7z$ in $D=\{z\in\mathbb{C}:|z|<1\}=D(0,1)$.

15) Find the number of zeros of $h(z)=\frac{1}{2}z^6-5z^4+7z$ in $D=\{z\in\mathbb{C}:1<|z|<3\}$.

16) Let $m\geq 2$ Find the number of zeros (including multiplicity) in $D=\{z\in\mathbb{C}:|z|<1\}=D(0,1)$

of $h(z)=ze^{m-z}-1$.

17) Let $D=\{z\in\mathbb{C}:|z|<1\}=D(0,1)$ and let $a_1,\dots,a_n\in D$ be all different.

Denote $B(z)=\prod_{j=1}^{n}\frac{z-a_j}{1-z\overline{a_j}}=\frac{z-a_1}{1-z\overline{a_1}}\cdot\ldots\cdot\frac{z-a_n}{1-z\overline{a_n}}$ and let $z_0\in D$.

Prove that the function $B(z)-z_0$ has n roots in D (including multiplicities).

Hint: show that $\varphi_j(z)=\frac{z-a_j}{1-z\overline{a_j}}$ maps the unit circle into itself.

18) Prove that there exists a number $N\in\mathbb{N}$ such that, for all $n>N$ $(n\in\mathbb{N})$, there is exactly one solution of the equation $\sin z=z^n$ in the disk $D=D(0,\frac{1}{2})$.

19) Let f be analytic on the closed unit disk $\overline{D}(0,1)$ and satisfy $|f(z)|<1$ for all $|z|=1$.

Prove that f has exactly one stationary point in the open unit disk. In other words,

prove that the equation $f(z)=z$ has exactly one solution in $D(0,1)$.

Hurwitz's Theorem

20) Prove that there exists $N\in\mathbb{N}$ such that, for all $n>N$, the equation

$z-\frac{z^3}{3!}+\frac{z^5}{5!}-\ldots+(-1)^{n+1}\frac{z^{2n+1}}{(2n+1)!}=0$ has exactly one solution in

$D=\{z:|z-\pi|<1\}$.

Hint: Use Hurwitz's theorem and the Taylor series for $\sin z$.

21) Let $R>0$. Prove that there exists $N\in\mathbb{N}$ such that, for all $n>N$, the equation

$1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\ldots+\frac{z^n}{n!}=0$ has no solutions in $D=D(0,R)=\{z:|z|<R\}$.

Hint: Use Hurwitz's theorem and the Taylor series for e^z.

Hurwitz's Theorem

22) Show that the function $f(z)=\dfrac{z-1}{z+1}$ has an analytic logarithm in $D=\mathbb{C}-\{(-\infty,-1]\cup[1,\infty)\}$.

23) Let D be the domain $\mathbb{C}-[-i,i]$, as illustrated.

a) Prove that the function $f(z)=\dfrac{z-i}{z+i}$ has an analytic logarithm on D.

b) Prove that the function $F(z)=z^2\dfrac{z-i}{z+i}$ has an analytic square root on D.

24) Let D be the domain $\mathbb{C}-[-1,1]$, and let $f(z)=1-z^2$ on D.

a) Prove that $f(z)$ does **not** have an analytic logarithm on D.

b) Prove that $f(z)$ **does** have an analytic square root on D.

25) Let D be the domain $\{z:|z|>4\}$, and let $f(z)=(z-2)^2-4$ on D.

a) Prove that $f(z)$ does **not** have an analytic logarithm on D.

b) Prove that $f(z)$ **does** have an analytic square root on D.

Answer Key :

1) $4\pi i$

2) $-2\pi i$

3) $12\pi i$

4) a) 0 is a pole order 5 and $\pm\pi i$ are poles of order 2.

b) -7

5) a) $z=(2k+1)\pi \ \ (k\in\mathbb{Z})$ are simple poles.

b) -5

6) -1

7) $N=1$

8) $N=2$

9) 1

10) 4

11) 4

12) 4

13) 3

14) 1

15) 3

16) 1

17) (proof)

18) (proof)

19) (proof)

20) (prove)

21) (prove)

22) (show)

23) a) (prove)

b) (prove)

24) a) (prove)

b) (prove)

25) a) (prove)

b) (prove)