

Workbook



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The Limit of a Function

Technique 1 – Substitution

Questions

Find the following limits if possible:

1) $\lim_{x \rightarrow 4} x^2 + x + 1$ 2) $\lim_{x \rightarrow 10} \frac{x+1}{x+2}$ 3) $\lim_{x \rightarrow 1^+} \sqrt{x+3}$ 4) $\lim_{x \rightarrow 100} 20$

Answer Key

1) 21

2) $\frac{11}{12}$

3) 2

4) 20

Technique 2 – Factoring

Questions

Find the following limits if possible:

1) $\lim_{x \rightarrow 3} \frac{x^2 - x - 6}{x^2 - 9}$

2) $\lim_{x \rightarrow -5} \frac{2x^2 - 50}{2x^2 + 3x - 35}$

3) $\lim_{x \rightarrow 1} \frac{x^7 - x}{x - 1}$

4) $\lim_{x \rightarrow 1} \frac{x^n - x}{x - 1} \quad n > 1$

Answer Key

1) $\frac{5}{6}$

2) $\frac{20}{17}$

3) 6

4) $n-1$

Technique 3 – Multiplication by Conjugate

Questions

Calculate the following limits if possible:

1) $\lim_{x \rightarrow 1} \frac{1 - \sqrt{x}}{1 - x}$

2) $\lim_{x \rightarrow 3} \frac{x - 3}{\sqrt{x + 1} - 2}$

3) $\lim_{x \rightarrow 3} \frac{3 - \sqrt{x + 6}}{2x - 6}$

4) $\lim_{x \rightarrow 1} \frac{\sqrt{x^2 + x + 2} - 2}{x^2 - 1}$

5) $\lim_{x \rightarrow 4} \frac{\sqrt{2x + 1} - \sqrt{x + 5}}{x - 4}$

6) $\lim_{x \rightarrow 1} \frac{2 - \sqrt{3x + 1}}{1 - \sqrt{x - 1}}$

7) $\lim_{x \rightarrow 1} \frac{1 - \sqrt[3]{x}}{1 - x}$

Answer Key

1) $\frac{1}{2}$

2) 4

3) $-\frac{1}{12}$

4) $\frac{3}{8}$

5) $\frac{1}{6}$

6) $\frac{3}{4}$

7) $\frac{1}{3}$

Technique 4 – $f(x)$ Tends to Infinity

Questions

Calculate the following limits if possible:

1) $\lim_{x \rightarrow 0} \frac{x^2 + 4}{x}$

2) $\lim_{x \rightarrow 2} \frac{(x-1)^2}{x-2}$

3) $\lim_{x \rightarrow 2} \frac{x^2 - 1}{(x-2)(x-5)}$

4) $\lim_{x \rightarrow 0^+} \frac{\ln x}{x}$

5) $\lim_{x \rightarrow 2^-} -\frac{1}{2} \ln(2-x)$

6) $\lim_{x \rightarrow 0^+} \left((\ln x)^2 + x(\ln x - 3) \right)$

7) $\lim_{x \rightarrow 0} e^{\frac{1}{x}}$

8) $\lim_{x \rightarrow 0^+} \frac{1}{1 + 2^{\frac{1}{x}}}$

9) $\lim_{x \rightarrow 0^-} \frac{1}{1 + 2^{\frac{1}{x}}}$

10) $\lim_{x \rightarrow 0} \frac{1}{1 + 2^{\frac{1}{x}}}$

11) $\lim_{x \rightarrow 0^+} \ln x \cdot \cot x$

Answer Key

- 1) No limit 2) No limit 3) No limit 4) $-\infty$ 5) ∞ 6) ∞
7) No limit 8) 0 9) 1 10) No limit 11) $-\infty$

Technique 5 – x Tends to Infinity

Questions

Find the following limits if possible:

- 1) $\lim_{x \rightarrow \infty} (e^{-x})^{\ln x}$
- 2) $\lim_{x \rightarrow -\infty} \arctan(x) + e^x$
- 3) $\lim_{x \rightarrow \infty} \frac{4x^2 + 2}{x^2 + 1000x}$
- 4) $\lim_{x \rightarrow -\infty} \frac{x^4 + 2x^2 + 6}{3x^2 + 10x}$
- 5) $\lim_{x \rightarrow \infty} \frac{x^4 + 2x^2 + 6}{3x^5 + 10x}$
- 6) $\lim_{x \rightarrow \infty} \left(\frac{x^2 - 5x + 6}{2x + 10} - \frac{x}{2} \right)$
- 7) $\lim_{x \rightarrow \infty} \frac{\sqrt{x^2 + 1}}{x}$
- 8) $\lim_{x \rightarrow -\infty} \frac{\sqrt{x^2 + 1}}{x}$
- 9) $\lim_{x \rightarrow -\infty} \frac{\sqrt{9x^6 - 5x}}{x^3 - 2x^2 + 1}$
- 10) $\lim_{x \rightarrow \infty} \frac{\sqrt[3]{x^4 + 2x^2 + 6 + 27x^6}}{\sqrt{3x^3 + 10x + 4x^4}}$
- 11) $\lim_{x \rightarrow \infty} \frac{\sqrt{x+2} - \sqrt{3x-3}}{\sqrt{4x+1} - \sqrt{5x-1}}$
- 12) $\lim_{x \rightarrow -\infty} \frac{16^x + 4^{x+1}}{2^{4x+2} + 2^{x+3}}$
- 13) $\lim_{x \rightarrow \infty} \frac{4 \cdot 9^x + 3^{x+1}}{81^{0.5x} + 3^{x+3}}$
- 14) $\lim_{x \rightarrow -\infty} \frac{4 \cdot 9^x + 3^{x+1}}{81^{0.5x} + 3^{x+3}}$
- 15) $\lim_{x \rightarrow \infty} \sqrt{\frac{4x^2 + 2}{x^2 + 1000x}}$
- 16) $\lim_{x \rightarrow \infty} \ln \left(\frac{3x^3 - 5x - 1}{x^3 - 2x^2 + 1} \right)$
- 17) $\lim_{x \rightarrow \infty} e^{\frac{x^4 + 2x^2 + 6}{3x^4 + 10x}}$
- 18) $\lim_{x \rightarrow -\infty} \sin \left(\frac{x^4 + 2x^2 + 6}{3x^5 + 10x} \right)$
- 19) $\lim_{x \rightarrow \infty} (\sqrt{x^2 + kx} - x)$
- 20) $\lim_{x \rightarrow \infty} (\sqrt{x^2 + x + 1} - x)$
- 21) $\lim_{x \rightarrow -\infty} (\sqrt{x^2 + x + 1} + x)$
- 22) $\lim_{x \rightarrow \infty} (\sqrt{x^4 + x^2 + 1} - x^2)$
- 23) $\lim_{x \rightarrow \infty} (\sqrt{x^2 + ax} - \sqrt{x^2 + bx})$

Answer Key

- | | | |
|-------------------|-------------------------------------|--------------------|
| 1) 0 | 2) $-\frac{\pi}{2}$ | 3) 4 |
| 4) $-\infty$ | 5) 0 | 6) -5 |
| 7) 1 | 8) -1 | 9) -3 |
| 10) $\frac{3}{2}$ | 11) $\frac{1-\sqrt{3}}{2-\sqrt{3}}$ | 12) 0 |
| 13) 4 | 14) $\frac{1}{9}$ | 15) 2 |
| 16) $\ln 3$ | 17) $e^{\frac{1}{3}}$ | 18) 0 |
| 19) $\frac{k}{2}$ | 20) $\frac{1}{2}$ | 21) $-\frac{1}{2}$ |
| 22) $\frac{1}{2}$ | 23) $\frac{a-b}{2}$ | |

Technique 6 – Euler's Limit

Questions

Find the following limits if possible:

1) $\lim_{x \rightarrow -\infty} \left(1 + \frac{1}{2x}\right)^x$

2) $\lim_{x \rightarrow -\infty} \left(1 + \frac{1}{x^2}\right)^x$

3) $\lim_{x \rightarrow \infty} \left(\frac{x+2}{x}\right)^x$

4) $\lim_{x \rightarrow \infty} \left(1 - \frac{1}{x^2}\right)^{x^2-1}$

5) $\lim_{x \rightarrow \infty} \left(\frac{2x+3}{2x-3}\right)^x$

6) $\lim_{x \rightarrow 0} (1 + \sin x)^{\frac{1}{x}}$

7) $\lim_{x \rightarrow \infty} \left(\frac{2x+3}{2x-3}\right)^x$

8) $\lim_{x \rightarrow \infty} \left(\frac{x^2 + x + 1}{x^2 + x + 4}\right)^{4x^2}$

9) $\lim_{x \rightarrow \infty} \left(1 + \tan \frac{1}{x}\right)^x$

Answer Key

1) \sqrt{e}

2) 1

3) e^2

4) e^{-1}

5) e^3

6) e

7) e^3

8) e^{-12}

9) e

Technique 7 – Trigonometric Limits

Questions

Find the following limits if possible:

1) $\lim_{x \rightarrow 0} \frac{\sin(3x)}{4x}$

2) $\lim_{x \rightarrow 0} \frac{\sin(3x)}{\sin(4x)}$

3) $\lim_{x \rightarrow 0} \frac{x \cos(x)}{\sin(2x)}$

4) $\lim_{x \rightarrow 0} \frac{1 - \cos(x)}{x^2}$

5) $\lim_{x \rightarrow 0} \frac{\tan(x) \sin(x)}{x}$

6) $\lim_{x \rightarrow 0} \frac{\sqrt{1 - \sin(x)} - \sqrt{\cos(x)}}{x}$

7) $\lim_{x \rightarrow 0} \frac{1 - \cos(1 - \cos(x))}{x^4}$

8) $\lim_{x \rightarrow 0} \frac{3 \sin(x) - \sin(3x)}{x^3}$

9) $\lim_{x \rightarrow 0} \frac{1 - \sqrt{\cos(x)}}{x^2}$

Answer Key

1) $\frac{3}{4}$

2) $\frac{3}{4}$

3) $\frac{1}{2}$

4) $\frac{1}{2}$

5) $\frac{1}{2}$

6) $\frac{1}{2}$

7) $\frac{1}{8}$

8) 4

9) $\frac{1}{4}$

Technique 8 – The Sandwich/Squeeze Theorem

Questions

Find the following limits if possible:

1) $\lim_{x \rightarrow \infty} \frac{\sin x}{x}$

2) $\lim_{x \rightarrow \infty} \frac{\cos(2x+1)}{x}$

3) $\lim_{x \rightarrow \infty} \frac{3x + \sin x}{4x + \cos x}$

4) $\lim_{x \rightarrow \infty} \frac{3x^2 + x + \sin 2x}{x^2 + \cos 3x}$

5) $\lim_{x \rightarrow 0} x \cdot \sin\left(\frac{1}{x}\right)$

6) $\lim_{x \rightarrow 0} x^2 \cdot \cos(\ln x^2)$

7) $\lim_{x \rightarrow \infty} \frac{3x + \arctan(2x - 3)}{4x + \arctan(x - \ln x)}$

8) $\lim_{x \rightarrow \infty} \sqrt[x]{2^x + 3^x + 4^x}$

9) $\lim_{x \rightarrow \infty} \frac{1}{x} [x]$

Answer Key

- | | | |
|------------------|------|------------------|
| 1) 0 | 2) 0 | 3) $\frac{3}{4}$ |
| 4) 3 | 5) 0 | 6) 0 |
| 7) $\frac{3}{4}$ | 8) 4 | 9) 1 |

Technique 9 – Piecewise Functions

Questions

Find the following limits if possible:

$$1) \lim_{x \rightarrow 0} f(x); f(x) = \begin{cases} \frac{\sin 4x}{x} & x > 0 \\ 4 + e^{\frac{1}{x}} & x < 0 \end{cases}$$

$$2) \lim_{x \rightarrow 1} f(x); f(x) = \begin{cases} \frac{x^2 + x - 2}{x - 1} & x > 1 \\ \frac{x - 1}{\sqrt{x} - 1} & x < 1 \end{cases}$$

$$3) \lim_{x \rightarrow 0} \frac{|x|}{x}$$

$$4) \lim_{x \rightarrow \infty} \frac{|x|}{x}$$

$$5) \lim_{x \rightarrow -\infty} \frac{|x|}{x}$$

Answer Key

- 1) 4
- 2) No limit
- 3) No limit
- 4) 1
- 5) -1

Limit from Definition

Questions

1) Use the definition of the limit to prove that $\lim_{x \rightarrow 2} (7x + 14) = 28$.

2) Use the definition of the limit to prove that $\lim_{x \rightarrow 3} x^2 = 9$.

3) Use the definition of the limit to prove that $\lim_{x \rightarrow 1} (x^2 - 1) = 0$.

4) Use the definition of the limit to prove that $\lim_{x \rightarrow 24} \sqrt{x+1} = 5$.

5) Use the definition of the limit to prove that $\lim_{x \rightarrow 1} \frac{1}{x} = 1$.

6) Use the definition of the limit to prove that $\lim_{x \rightarrow \frac{\pi}{4}} \sin x = \sin \frac{\pi}{4}$.

7) Use the definition of the limit to prove that $\lim_{x \rightarrow 2} \frac{3+x}{x^2+1} = 1$.

8) Use the definition of the limit to prove that $\lim_{x \rightarrow 4^-} (\sqrt{4-x}) = 0$.

9) Use the definition of the limit to prove that:

a. $\lim_{x \rightarrow 0^+} \frac{|x|}{x} = 1$

b. $\lim_{x \rightarrow 0^-} \frac{|x|}{x} = -1$

10) Use the definition of the limit to prove that $\lim_{x \rightarrow 2} \frac{-5}{(x-2)^2} = -\infty$.

11) Use the definition of the limit to prove that $\lim_{x \rightarrow 3^-} \frac{-2}{x-3} = \infty$.

12) Use the definition of the limit to prove that $\lim_{x \rightarrow 0^+} \ln x = -\infty$.

13) Use the definition of the limit to prove that $\lim_{x \rightarrow \infty} \frac{x+7}{x+2} = 1$.

14) Use the definition of the limit to prove that $\lim_{x \rightarrow \infty} \frac{3-4x}{2x+1} = -2$.

15) Use the definition of the limit to prove that $\lim_{x \rightarrow \infty} \frac{3x^2-1}{x^2+x+1} = 3$.

16) Given a function $f(x)$ which satisfies $\lim_{x \rightarrow \infty} f(x) = -5$.

Prove that there exists an $M > 0$ such that $f(x) < -4$ whenever $x > M$.

17) Given a function $f(x)$ which satisfies $\lim_{x \rightarrow -\infty} f(x) = 5$.

Prove that there exists an $M < 0$ such that $f(x) > 4$ whenever $x < M$.

18) Given a positive function $f(x)$ on the interval $[a, \infty)$ which satisfies $\lim_{x \rightarrow \infty} f(x) = 0$.

Prove [using *epsilon-delta*] that $\lim_{x \rightarrow \infty} \sqrt{f(x)} = 0$.

19) Given the limit: $\lim_{x \rightarrow \infty} \frac{x^2+2x}{\underbrace{x^2+3x+2}_{f(x)}} = 1$ (*no need to prove*).

Find a value for $M > 0$ such that $|f(x) - L| < 0.1$ whenever $x > M$.

20) Use the definition of continuity to prove that $f(x) = 2x - 3$ is continuous at $x = 4$.

Answer Key:

1-18) and **20)** Refer to the videos.

19) $x > 30$