Workbook

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Vector Spaces over R (Rn)

The Vector Space Rn

Questions

1) Check if W is a subspace of $\mathbb{R}^3$, where $W=\{\langle a,b,c\rangle \mid a+b+c=0\}$.

2) Check if W is a subspace of $\mathbb{R}^3$, where $W=\{\langle a,b,c\rangle \mid a=c\}$.

3) Check if W is a subspace of $\mathbb{R}^3$, where $W=\{\langle a,b,c\rangle \mid a=3b\}$.

4) Check if W is a subspace of $\mathbb{R}^3$, where $W=\{\langle a,b,c\rangle \mid a<b<c\}$.

5) Check if W is a subspace of $\mathbb{R}^3$, where $W=\{\langle a,b,c\rangle \mid a=c^2\}$.

6) Check if W is a subspace of $\mathbb{R}^3$, where $W=\{\langle a,b,c\rangle \mid c-b=b-a\}$.

7) Check if W is a subspace of $\mathbb{R}^3$, where $W=\{\langle a,b,c\rangle \mid b=a\cdot q,\ c=a\cdot q^2\}$.

Answer Key

1) W is a subspace of $\mathbb{R}^3$.

2) W is a subspace of $\mathbb{R}^3$.

3) W is a subspace of $\mathbb{R}^3$.

4) W is not a subspace of $\mathbb{R}^3$.

5) W is not a subspace of $\mathbb{R}^3$.

6) W is a subspace of $\mathbb{R}^3$.

7) W is not a subspace of $\mathbb{R}^3$.

Linear Combination, Dependence and Span

Questions

1) We are given the following vectors in $\mathbb{R}^4$:
$u_1 = \langle 4,1,1,5 \rangle$, $u_2 = \langle 0,11,-5,3 \rangle$, $u_3 = \langle 2,-5,3,1 \rangle$, $u_4 = \langle 1,3,-1,2 \rangle$.

a. Is u_1 a linear combination of u_4?
b. Does u_1 belong to $Sp\{u_4\}$?
c. Is the set $\{u_1, u_4\}$ linearly dependent?

2) We are given the following vectors in $\mathbb{R}^4$:
$u_1 = \langle 4,1,1,5 \rangle$, $u_2 = \langle 0,11,-5,3 \rangle$, $u_3 = \langle 2,-5,3,1 \rangle$, $u_4 = \langle 1,3,-1,2 \rangle$.

a. Is u_3 a linear combination of u_1 and u_2?
b. Does u_3 belong to $Sp\{u_1, u_2\}$?
c. Is the set $\{u_1, u_2, u_3\}$ linearly dependent?
If so, try to write each vector in the set as a linear combination of the others.

3) We are given the following vectors in $\mathbb{R}^4$:
$u_1 = \langle 4,1,1,5 \rangle$, $u_2 = \langle 0,11,-5,3 \rangle$, $u_3 = \langle 2,-5,3,1 \rangle$, $u_4 = \langle 1,3,-1,2 \rangle$.

a. Is u_4 a linear combination of u_1 and u_2?
b. Does u_4 belong to $Sp\{u_1, u_2\}$?
c. Is the set $\{u_1, u_2, u_4\}$ linearly dependent?
If so, try to write each vector in the set as a linear combination of the others.

4) We are given the following vectors in $\mathbb{R}^4$:
$u_1 = \langle 4,1,1,5 \rangle$, $u_2 = \langle 0,11,-5,3 \rangle$, $u_3 = \langle 2,-5,3,1 \rangle$, $u_4 = \langle 1,3,-1,2 \rangle$,
and $v = \langle 4,12,k,-2k \rangle$.

a. What should the value of k be, in order for the vector v to be a linear combination of u_1 and u_2?
b. What should the value of k be, in order for the vector v to belong to $Sp\{u_1, u_2\}$?
c. What should the value of k be in order for the set $\{u_1, u_2, v\}$ to be linearly dependent?

5) We are given the following vectors in $\mathbb{R}^4$:
$u_1 = \langle 4,1,1,5 \rangle, \quad u_2 = \langle 0,11,-5,3 \rangle, \quad u_3 = \langle 2,-5,3,1 \rangle, \quad u_4 = \langle 1,3,-1,2 \rangle,$
and $v = \langle a,b,c,d \rangle$.

a. What are the conditions on a,b,c,d, in order for v to be a linear combination of u_1 and u_2?
b. What are the conditions on a,b,c,d, in order for v to belong to $Sp\{u_1,u_2\}$?
c. What are the conditions on a,b,c,d, in order for the set $\{u_1,u_2,v\}$ to be linearly dependent?

6) Check if the following set of vectors is linearly dependent:
$A = \langle u =< 1,2,3 >, v =< 4,5,6 >, w < 7,8,9 > \rangle$

7) Check if the following set of vectors is linearly dependent:
$A = \langle u =< 1,2,3 >, v =< 4,5,6 >, w < 7,8,10 > \rangle$

Answer Key

1) a. No b. No c. No

2) a. Yes. $u_3 = \frac{1}{2}(u_1 - u_2)$ b. Yes. Follows from a. c. Yes $u_1 = 2u_3 + u_2$, $u_2 = u_1 - 2u_3$

3) a. Yes. $u_4 = \frac{1}{4}(u_1 + u_2)$ b. Yes. Follows from a. c. Yes $u_1 = 4u_4 - u_2$, $u_2 = 4u_4 - u_1$

4) a.b.c. $k = -4$

5) a. Refer to the video. b. Refer to the video. c. $\begin{cases} a = 5t + 3s \\ b = 4t - 13s \\ c = 7s \\ d = 7t \end{cases}$

6) $w = 2v - u$

7) A is linearly **independent**.

Basis for Rn

Questions

1) Check if each of the following sets is a basis of $\mathbb{R}^3$:

a. $\{\langle 1,0,1\rangle,\langle 0,0,1\rangle\}$

b. $\{\langle 1,1,2\rangle,\langle 1,2,3\rangle,\langle 3,3,4\rangle,\langle 2,2,1\rangle\}$

c. $\{\langle 1,2,3\rangle,\langle 4,5,6\rangle,\langle 7,8,9\rangle\}$

2) Check if each of the following sets is a basis of $M_{2\times 2}[\mathbb{R}]$ (AKA $M_2[\mathbb{R}]$):

a. $\left\{\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix},\begin{bmatrix}5 & 6\\ 7 & 8\end{bmatrix},\begin{bmatrix}9 & 1\\ 2 & 3\end{bmatrix}\right\}$

b. $\left\{\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix},\begin{bmatrix}5 & 6\\ 7 & 8\end{bmatrix},\begin{bmatrix}9 & 1\\ 2 & 3\end{bmatrix},\begin{bmatrix}5 & 6\\ 7 & 2\end{bmatrix},\begin{bmatrix}5 & 16\\ 7 & 8\end{bmatrix}\right\}$

c. $\left\{\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix},\begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix},\begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix},\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right\}$

3) Check if each of the following sets is a basis of $P_2[\mathbb{R}]$ (deg ≤ 2 poly):

a. $\{1+x,\ x^2+2x+3\}$

b. $\{1+x,\ x^2+2x+3, 2x+4x^2, x-x^2\}$

c. $\{1+2x+3x^2,\ 4+5x+6x^2, 7+8x+10x^2\}$

4) Consider the following set of vectors in $\mathbb{R}^3$: $T=\{\langle 1,2,3\rangle,\ \langle 4,5,6\rangle,\ \langle 7,8,9\rangle,\langle 2,3,4\rangle\}$.

a. Is T a basis of $\mathbb{R}^3$?

b. Find a maximal linearly independent subset T' of T.

c. Extend T to a basis of $\mathbb{R}^3$.

Answer Key

1) a. The two vectors can't form a basis for $\mathbb{R}^3$.

b. The four vectors can't form a basis for $\mathbb{R}^3$.

c. The vectors are linearly dependent and so are not a basis for $\mathbb{R}^3$.

2) a. The three vectors can't form a basis for $M_{2\times2}[\mathbb{R}]$.

b. The five vectors can't form a basis for $M_{2\times2}[\mathbb{R}]$.

c. The four vectors do form a basis for $M_{2\times2}[\mathbb{R}]$.

3) a. The two vectors can't form a basis for $P_2[\mathbb{R}]$.

b. The four vectors can't form a basis for $P_2[\mathbb{R}]$.

c. The three vectors do form a basis for $P_2[\mathbb{R}]$.

4) a. No, since it contains more than three vectors.

b. $T' = \{\langle 1,2,3\rangle, \langle 4,5,6\rangle\}$

c. $T^* = \{\langle 1,2,3\rangle, \langle 4,5,6\rangle, \langle 0,0,1\rangle\}$

Solution Space of Homogenous SLE

Questions

1) Here are 3 systems of linear equations (SLEs):

1. $\begin{cases} x+y-z+2w=0 \\ 3x-y+7z+4w=0 \\ -5x+3y-15z-6w=0 \end{cases}$ 2. $\begin{cases} x-y+z+w=0 \\ x+2z-w=0 \\ x+y+3z-3w=0 \end{cases}$ 3. $\begin{cases} x-y+z+w=0 \\ 2x-2y+2z+2w=0 \end{cases}$

Let's denote by W, U and V the subspaces spanned by 1., 2. and 3., respectively.

a. Find a basis for each of W, U and V as well as their dimension.

b. i. Find a basis for $U+V$ and its dimension.
   ii. Find the dimension of $U \cap V$.

c. Find a basis for $U \cap V$.

2) Let $U=\{\langle a,b,c,d\rangle \in \mathbb{R}^4 \mid a=c, b=d\}$. Find a basis and the dimension of U.

3) Let $U=\{\langle a,b,c,d\rangle \in \mathbb{R}^4 \mid c=a+b, d=b+c\}$. Find a basis and the dimension of U.

4) Let $U=\{v \in \mathbb{R}^4 \mid v \cdot \langle 1,-1,1,-1\rangle = 0\}$. Find a basis and the dimension of U.

Answer Key

1) a. $B_W = \{\langle -1.5, 2.5, 1, 0\rangle,\ \langle -1.5, -0.5, 0, 1\rangle\}\ \dim W = 2$,

$B_U = \{\ \langle -2, -1, 1, 0\rangle,\ \langle 1, 2, 0, 1\rangle\ \}\ \dim U = 2$,

$B_V = \{\ \langle -1, 0, 0, 1\rangle,\ \langle -1, 0, 1, 0\rangle,\ \langle 1, 1, 0, 0\rangle\ \}\ \dim V = 3$

b. i. $B_{U+V} = \{\langle 0, 0, -1, 1\rangle,\ \langle 0, 1, 1, 0\rangle,\ \langle 1, 1, 0, 0\rangle\}\ \dim U + V = 3$

b. ii. $2 = \dim(U \cap V)$

c. $B_{U \cap V} = \{\langle -2, -1, 1, 0\rangle,\ \langle 1, 2, 0, 1\rangle\}$

2) $B_U = \{\langle 0, 1, 0, 1\rangle, \langle 1, 0, 1, 0\rangle\}$, $\dim U = 2$

3) $B_U = \{\ \langle -1, 1, 0, 1\rangle, \langle 2, -1, 1, 0\rangle\}$, $\dim U = 2$

4) $B_U = \{\ \langle 1, 0, 0, 1\rangle, \langle -1, 0, 1, 0\rangle, \langle 1, 1, 0, 0\rangle\}\ \dim U = 3$

Subspaces

Questions

1) Consider the subspace of $\mathbb{R}^4$ defined as follows:
$U = span\{\langle 1,1,-1,2\rangle, \langle 3,-1,7,4\rangle, \langle -5,3,-15,-6\rangle\}$.

a. Find a basis for U and its dimension.
b. Find a homogeneous SLE whose solution space is U.

2) Consider the subspace of $\mathbb{R}^4$ defined as follows:
$V = span\{\langle 1,1,-1,1\rangle, \langle 1,0,2,-1\rangle, \langle 1,1,3,-3\rangle, \langle 5,1,5,8\rangle\}$.

Find, for V : a basis, the dimension, a homogeneous SLE.

3) Consider the two subspaces of $\mathbb{R}^4$ defined as follows:
$U = span\{\langle 1,1,-1,2\rangle, \langle 3,-1,7,4\rangle, \langle -5,3,-15,-6\rangle\}$
$V = span\{\langle 1,-1,1,1\rangle, \langle 1,0,2,-1\rangle, \langle 1,1,3,-3\rangle, \langle 5,1,5,8\rangle\}$.

Find a basis and the dimension of $U+V$.

4) Consider the two subspaces of $\mathbb{R}^4$ defined as follows:
$U = span\{\langle 1,1,-1,2\rangle, \langle 3,-1,7,4\rangle, \langle -5,3,-15,-6\rangle\}$
$V = span\{\langle 1,-1,1,1\rangle, \langle 1,0,2,-1\rangle, \langle 1,1,3,-3\rangle, \langle 5,1,5,8\rangle\}$.

Find a basis and the dimension of $U \cap V$.

Answer Key

1) a. $B_U=\{\langle 1,1,-1,2\rangle,\langle 0,-4,10,-2\rangle\}$, $\dim U=2$ b. $\begin{cases}-3x+5y+2z=0\\-3x-y+2t=0\end{cases}$

2) $-8x-y+5z+2t=0$, $B_V=\{\langle 1,-1,1,1\rangle,\langle 0,1,1,-2\rangle,\langle 0,0,-2,5\rangle\}$, $\dim V=3$

3) $B_{U+V}=\{\langle 1,1,-1,2\rangle,\langle 0,-4,10,-2\rangle,\langle 0,0,1,0\rangle,\langle 0,0,0,1\rangle\}$, $\dim U+V=4$

4) $\dim U\cap V=1$, $B_{U\cap V}=\{\langle 5,1,5,8\rangle\}$

Row and Column Space

Questions

1) Find a basis and the dimension of the row space and the column space of the following matrix. What is its rank?

$$\begin{bmatrix} 4 & 1 & 1 & 5 \\ 0 & 11 & -5 & 3 \\ 2 & -5 & 3 & 1 \\ 1 & 3 & -1 & 2 \end{bmatrix}$$

2) Find a basis and the dimension of the row space and the column space of the following matrix. What is its rank?

$$\begin{bmatrix} 1 & 2 & 1 & 3 & 5 \\ 2 & 5 & 3 & 1 & 6 \\ 1 & -1 & -2 & 2 & 1 \\ -2 & 3 & 5 & -4 & -1 \end{bmatrix}$$

Answer key

1) $B_{row}=\{\langle 4,1,1,5\rangle,\langle 0,11,-5,3\rangle\}$ $\dim(row)=2$;

$$B_{col}=\left\{\begin{bmatrix}4\\0\\2\\1\end{bmatrix},\begin{bmatrix}0\\4\\-2\\1\end{bmatrix}\right\},\ \dim(col)=2$$;rank=2.

2) $B_{\text{row}}=\{\langle 1,2,1,3,5\rangle,\langle 0,1,1,-5,-4\rangle,\langle 0,0,0,1,1\rangle\}$; $\dim(row)=3$.

$$B_{\text{col}}=\left\{\begin{bmatrix}1\\2\\1\\-2\end{bmatrix},\begin{bmatrix}0\\1\\-3\\7\end{bmatrix},\begin{bmatrix}0\\0\\-16\\37\end{bmatrix}\right\};\ \dim(col)=3$$; Rank=3.