

Workbook



Table of Contents

Tangents, Normal Lines and Linear Approximation	2
Tangent and Normal Lines – Basic Exercises	2
Tangent and Normal Lines – Exercises with a Constant	5
Tangent and Normal Lines of Implicit Functions	7
Tangent and Normal Lines – Parametric	8
The Angle Between Two Curves	9
Vertical Tangents and Cusps	10
Linear Approximation	12

Tangents, Normal Lines and Linear Approximation

Tangent and Normal Lines – Basic Exercises

Questions

- 1) Find the equation of the line that is tangent to the curve $f(x) = x^2 - 2x + 3$ at the point on the curve where $x = 2$.
- 2) Find the equation of the line that is normal to the curve $f(x) = x^2 - 2x + 3$ at the point on the curve where $x = 2$.
- 3) Find the equation of the line that is tangent to the curve $f(x) = x^3 - 4x + 2x - 5$ at the point on the curve where $x = 1$, Does the line intersect the curve at any other point?
- 4) Find the equation of the line that is tangent to the curve $f(x) = x^3 - 6x + 2$ and are parallel to the line $y = 6x - 2$.
- 5) Find the equation of the line that is tangent to the curve $f(x) = y = \sqrt{2x + 3}$ at the point on the curve where $x = 3$.
- 6) Find the equation of the line that is tangent to the curve $f(x) = xe^{x^2}$ at the origin. Does the line intersect the curve at any other point?
- 7) Find the equation of the tangent to the curve $f(x) = e^{\sin(4x)}$ at the point on the curve where $x = \pi$.
- 8) Find the equation of the lines that are normal to the curve $f(x) = \frac{2x}{1-x}$ and parallel to the line $y = -2x$.

- 9) Find the equation of the line that is tangent to the curve $f(x) = x \ln x$ at the point on the curve where $x = e$.
- 10) Find the equation of the line that is tangent to the curve $f(x) = e^{x-1} \ln x$ at the point on the curve where $x = 1$.
- 11) Find the equation of the tangent to the curve $f(x) = \sin 2x$ at the point on the curve where $x = \frac{\pi}{2}$.
- 12) At how many different values of x does the curve $y = x^3 - 2x + 1$ have a tangent line parallel to the line $y = x + 1$?
- 13) Find the equation of the tangent to $f(x) = x^{4x}$ at the point where $x = 1$.
- 14) Find the equation of the tangent to $f(x) = (2x + 1)^{x^2 + 1}$ at the point where $x = 0$.

Answer Key

1) $y = 2x - 1$

2) $y = -\frac{1}{2}x + 4$

3) $y = -3x - 3$, Intersections: $(1, -6), (2, -9)$

4) $y = 6x - 14$; $y = 6x + 18$

5) $y = \frac{1}{3}x + 4$

6) $y = x$, No additional intersection points.

7) $y = 4x - 4\pi + 1$

8) $y = -2x - 1$; $y = -2x + 3$

9) $y = 2x - e$

10) $y = x - 1$

11) $y = -2x + \pi$

12) 2 Points

13) $y = 4x - 3$

14) $y = 2x + 1$

Tangent and Normal Lines – Exercises with a Constant

Questions

- 1) Determine the constant k such that the line $y = -x + 3$ is tangent to the curve $y = \frac{k}{x+1}$.
- 2) Determine the constant n such that the line $y = -x + n$ is tangent to the curve $y = \frac{1}{x}$.
- 3) Determine the constants a, c such that the line $y = ax + 0.5$ is tangent to the curve $y = \frac{2}{x+c}$ at the point where $x = 0$.
- 4) Write the equation of the straight line passing through the point $(-1, 0)$ and tangent to the curve $f(x) = \sqrt{x}$. Sketch the curve and the line.
- 5) Write the equation of the straight lines passing through $(1.5, 0)$ and tangent to the curve $f(x) = \frac{1}{4}x + 1$, Sketch the curve and the lines and prove that the lines are perpendicular.
- 6) Find all lines that can be drawn tangent to the curve $f(x) = x^2 - 2x + 1$ from the point $(2, -3)$. Sketch the curve and the lines.
- 7) Determine the constant b such that the line $y = 3x$ is tangent to the curve $y = x\sqrt{x} + b$.
- 8) Determine the constant c such that the curve $y = -0.5x^2 + c$ is tangent to the curve $y = \frac{1}{x}$, Find the tangent point and the joint tangent.
- 9) Determine the constant c such that the parabola $y = -x^2 + c$ is tangent to the parabola $y = x^2 - 4x + 6$, Find the tangent point and the joint tangent.
- 10) Determine the constant c such that the curve $y = -8x^2 + c$ is tangent to the curve $y = \frac{1+6x^2}{2x^2}$, Find the tangent point(s) and the joint tangents(s).

Answer Key

1) $k = 4$.

2) $n = \pm 2$.

3) $a = -\frac{1}{8}, c = 5$.

4) $y = \frac{1}{2}x + \frac{1}{2}$.

5) $y = 2x - 3, y = -\frac{1}{2}x + \frac{3}{4}$.

6) $y = 6x - 15, y = -2x + 1$.

7) $b = 4$.

8) $c = 1.5$, tangent point: $(1, 1)$, joint tangent: $y = x + 2$.

9) $c = 4$, tangent point: $(1, 3)$, joint tangent: $y = -2x + 5$.

10) $c = 7$, tangent point: $\left(\frac{1}{2}, 5\right)$, joint tangent: $y = -8x + 9$

$c = 7$, tangent point: $\left(-\frac{1}{2}, 5\right)$, joint tangent: $y = 8x + 9$.

Tangent and Normal Lines of Implicit Functions

Questions

- 1) Find the equation of the line tangent to $x^2 + y^2 = 2xy - 2x - y + 6$ at $(2, 2)$.
- 2) Find the equation of the line tangent to $x^3 + 4xy - 4y^3 = 1$. At the point (x, y) on the curve where $x = 1, y > 0$.
- 3) Find the equation of the line tangent to $\sqrt{x} + \sqrt{y} = 2\sqrt{a}$, $a > 0$, at the point on the curve where $x = a$.

Answer Key

1) $y = -2x + 6$

2) $y = -\frac{1}{11}x - \frac{10}{11}$

3) $y = -x + 2a$

Tangent and Normal Lines – Parametric

Questions

- 1) Give the following function $x = f(t)$, $y = g(t)$ where: $x = \sqrt{2t^2 + 1}$, $y = 4t^2 + 4t + 1$:
- Find $\frac{dy}{dx}$.
 - Find the equation of the line tangent to the curve at the point for which $t = 2$.
- 1) Give the following function $x = f(t)$, $y = g(t)$ where: $x = \sqrt{2t^3 + 5t^2}$, $y = \sqrt[3]{4t}$:
- Find $\frac{dy}{dx}$.
 - Find the equation of the line tangent to the curve at the point for which $t = 2$.

Answer Key

- 1) a. $\frac{dy}{dx} = \frac{(4t+2)(\sqrt{2t^2+1})}{t}$ b. $y = 15x - 20$.
- 2) a. $\frac{dy}{dx} = \frac{4\sqrt{2t^3+4t^2}}{3\sqrt[3]{(4t)^2 \cdot (3t^2+5t)}}$ b. $y = \frac{2}{\sqrt{22}}x - \frac{22 - 6\sqrt{22}}{11}$.

The Angle Between Two Curves

Questions

- 1) Show that the curve $x^2 + 2y^2 = 8$ and the curve $x^2 - y^2 = 2$ intersect at right angles.
- 2) Find the angles between the following pairs of curves:
 - a. $y = x^2, y = \frac{1}{x}$
 - b. $x^2 + y^2 = 8, y^2 = 2x$

Answer Key

- 1) Slope 1: $\frac{-1}{\sqrt{2}}$, Slope 2: $\sqrt{2} \cdot \frac{-1}{\sqrt{2}} \cdot \sqrt{2} = -1 \Rightarrow$ The curves are perpendicular
- 2) a. $\alpha = 71.57^\circ$ b. $\alpha = 71.57^\circ$

Vertical Tangents and Cusps

Questions

1) Answer the following questions:

- Find all the points on the graph $y = \sqrt[5]{4-x}$ where the tangent line is vertical.
- Does the function have a vertical cusp?

2) Answer the following questions:

- Find all the points on the graph $y = \sqrt{x} + \sqrt[3]{x}$ where the tangent line is vertical.
- Does the function have a vertical cusp?

3) Answer the following questions:

- Find all the points on the graph $y = \sqrt[3]{x^2}$ where the tangent line is vertical.
- Does the function have a vertical cusp?

4) Answer the following questions:

- Find all the points on the graph $y = |x^3 - 27|$ where the tangent line is vertical
- Does the function have vertical cusp?

5) Answer the following questions:

- Find all the points on the graph $y = \sqrt{4-x^2}$ where the tangent line is vertical.
- Does the function have vertical cusp?

6) Answer the following questions:

- Find all the points on the graph $f(x) = \begin{cases} x^{\frac{1}{3}} + 4 & x \leq 0 \\ 4 - x^{\frac{1}{5}} & x > 0 \end{cases}$ where the tangent line is vertical.
- Does the function have vertical cusp?

Answer Key

- | | |
|----------------------|--------|
| 1) a. $(4,0)$ | b. No |
| 2) a. $(2,0),(-2,0)$ | b. No |
| 3) a. $(0,0)$ | b. Yes |
| 4) a. None | b. No |
| 5) a. $(-2,0),(2,0)$ | b. No |
| 6) a. $(0,4)$ | b. Yes |

Linear Approximation

Questions

- 1) Use linear approximation to approximate the value of $\sqrt[5]{33}$.
- 2) Use linear approximation to approximate the value of $\sqrt[4]{15}$.
- 3) Use linear approximation to approximate the value of $\sin 3^\circ$.
- 4) Use linear approximation to approximate the value of $\arctan 0.25$.
- 5) Use linear approximation to approximate the value of $\frac{1}{e}$.

Answer Key

- 1) 2.0125 2) 1.96875 3) 0.05236 4) 0.25
5) 0