

Workbook



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Second Order Linear Equations

Missing x or y , Reduction of Order

Questions

Solve the following equations:

1) $x^2 y'' + xy' = \frac{1}{x} \quad (x \neq 0)$

2) $y'' \tan x - 1 = y', \quad (\cos x \neq 0)$

3) $2xy' y'' - (y')^2 + 1 = 0$

4) $y'' x \ln x = y'$

5) $xy'' = x^2 e^x + y'$

6) $y \cdot y'' + (y')^2 = 0$

7) $2y'' y - (y')^2 = 1$

8) $y'' \tan y = 2(y')^2 \quad (\cos y \neq 0)$

Answer Key

1) $y = \frac{1}{x} + C_1 \cdot \ln x + C_2$

2) $y = -x + C_1 \cdot \cos x + C_2$

3) $y = \pm x + C_3$

4) $y = C_3$

5) $y = e^x (x-1) + C_1 \frac{x^2}{2} + C_2$

6) $\frac{y^2}{2} = cx + k, y = c$

7) $y = \frac{1}{c} \left[\frac{c^2 (x+k)^2}{4} + 1 \right]$

8) $\cot y = -(cx + k), y = c$

Linear, Homogeneous, Constant Coefficients

Questions

Solve the following equations:

1) $y'' - 100y = 0$

2) $y'' - 4y' = 0$

3) $y'' - 8y' + 7y = 0$

4) $4z'' + z' - 5z = 0$; $z(0) = 1$, $z'(0) = 1$

5) $y'' - 2y' + y = 0$

6) $4\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + x = 0$

7) $y'' + 10y' + 125y = 0$

8) $y'' + 4y = 0$

9) $y'' - 2y' + 10y = 0$; $y(0) = 0$, $y'(0) = 3$

Answer Key

1) $y = c_1e^{10x} + c_2e^{-10x}$

2) $y = c_1 + c_2e^{4x}$

3) $y = c_1e^x + c_2e^{7x}$

4) $z = e^x$

5) $y = c_1e^x + c_2xe^x$

6) $x = c_1e^{-0.5t} + c_2te^{-0.5t}$

7) $y = e^{-5x} [c_1 \cos 10x + c_2 \sin 10x]$

8) $y = c_1 \cos 2x + c_2 \sin 2x$

9) $y = e^x \sin 3x$

Linear, Nonhomogeneous, Constant Coefficients - Method of Undetermined Coefficients

Questions

Solve the following equations:

- | | |
|--|--|
| 1) $y'' + 5y' + 6y = 22x + 6x^2$ | 2) $y'' - 2y' + y = e^{2x}; y(0) = 2, y'(0) = 7$ |
| 3) $y'' - y' - 2y = 4\sin 2x$ | 4) $y'' - 2y = xe^{-x}$ |
| 5) $y'' - y = 3e^{2x} \cos x$ | 6) $z'' + z = \sin x$ |
| 7) $y'' - 3y' + 2y = 2x^2 + e^x + 2xe^x + 4e^{3x}$ | 8) $y'' + 3y' = 9x$ |
| 9) $y'' - 3y' + 2y = e^x$ | 10) $y'' - 2y' = 6x^2 - 2x$ |
| 11) $x'' + 5x' + 6x = e^{-t} + e^{-2t}$ | 12) $y'' + 2y' + 5y = e^{-x} \sin 2x$ |
| 13) $y'' + 3y' - 4y = e^{3ix}$ | |

Answer Key

- | | |
|--|--|
| 1) $y = c_1 e^{-3x} + c_2 e^{-2x} + x^2 + 2x - 2$ | 2) $y = e^x + 4xe^x + e^{2x}$ |
| 3) $y = c_1 e^{2x} + c_2 e^{-x} + \frac{1}{5} \sin 2x - \frac{3}{5} \cos 2x$ | 4) $y = c_1 e^{-\sqrt{2}x} + c_2 e^{\sqrt{2}x} + (2-x)e^{-x}$ |
| 5) $y = c_1 e^{-x} + c_2 e^x + \frac{3}{10} e^{2x} \cos x + \frac{3}{5} e^{2x} \sin x$ | 6) $z = c_1 \cos x + c_2 \sin x - \frac{1}{2} x \cos x$ |
| 7) $y = c_1 e^x + c_2 e^{2x} + x^2 + 3x + 3.5 - x^2 e^x - 3x e^x + 2e^{3x}$ | 8) $y = c_1 + c_2 e^{-3x} + \frac{3}{2} x^2 - x$ |
| 9) $y = c_1 e^x + c_2 e^{2x} - x e^x$ | 10) $y = c_1 e^{-3x} + c_2 e^{-2x} - x^2 - x - x^3$ |
| 11) $x = c_1 e^{-2t} + c_2 e^{-3t} + \frac{1}{2} \cdot e^{-t} + t e^{-2t}$ | 13) $y = \left(\frac{-13}{250} - \frac{9}{250} i \right) e^{3ix}$ |
| 12) $y = e^{-x} [c_1 \cos 2x + c_2 \sin 2x] - \frac{1}{4} x \cdot e^{-x} \cos 2x$ | |

Linear, Nonhomogeneous, Constant Coefficients - Method of Variation of Parameters

Questions

Solve the following equations:

$$1) \quad y'' + y = \frac{1}{\sin x}$$

$$2) \quad y'' + 4y' + 4y = e^{-2x} \ln x$$

$$3) \quad y'' + 2y' + y = 3e^{-x} \sqrt{x+1}$$

$$4) \quad y'' - 2y' + y = \frac{e^x}{x} ; \quad y(1) = 0, \quad y'(1) = 0$$

$$5) \quad y'' - 3y' + 2y = \frac{1}{1+e^{-x}}$$

$$6) \quad y'' + 4y = \sec 2x$$

Answer Key

$$1) \quad y = c_1 \cos x + c_2 \sin x - \cos x \cdot x + \sin x \cdot \ln |\sin x|$$

$$2) \quad y = c_1 e^{-2x} + c_2 x e^{-2x} - e^{-2x} \frac{x^2}{2} \left[\ln x - \frac{1}{2} \right] + x^2 e^{-2x} [\ln x - 1]$$

$$3) \quad y = c_1 e^{-x} + c_2 x e^{-x} - e^{-x} \left[\frac{6(\sqrt{x+1})^5}{5} - \frac{6(\sqrt{x+1})^3}{3} \right] + x e^{-x} [2(x+1)^{3/2}]$$

$$4) \quad y = e^x - x e^x + x e^x \ln x, \quad (x > 0)$$

$$5) \quad y = c_1 e^x + c_2 e^{2x} + e^x \ln(1+e^{-x}) + e^{2x} [\ln(1+e^{-x}) - (1+e^{-x})]$$

$$6) \quad y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{4} \cos 2x \ln |\cos 2x| + \frac{1}{2} \sin 2x \cdot x$$

Linear, Homogeneous, Non-Constant Coefficients - 2nd Solution Method

Questions

Solve the following equations:

1) $y'' + \tan x \cdot y' - (2 \tan x + 4)y = 0$

2) $(1 - x^2)y'' + 2xy' - 2y = 0, (x > 0)$

3) $(1 + x^2)y'' - 1.5xy' + y = 0, (x > 2)$

Answer Key

1) $y = c_1 e^{2x} + c_2 e^{-2x} (\sin x - 4 \cos x)$

2) $y = c_1 x + c_2 (x^2 + 1)$

3) $y = c_1 (x^2 - 2) + c_2 (x^2 - 2)h(x)$ where $h(x)$ is defined in the recording.

The Wronskian and Its Uses

Questions

Solve the following equations:

- 1) Is it possible that $y_1(x) = e^x$, $y_2(x) = \sin x$ are two solutions of a linear homogeneous 2nd order ODE $y'' + p(x)y' + q(x)y = 0$, with continuous coefficients $p(x), q(x)$ on the interval $[0, \pi]$?

- 2) Answer the following questions:
 - a. Show that the functions $y_1(x) = \sin x^2$, $y_2(x) = \cos x^2$ are l.i. solutions of the equation $xy'' - y' + 4x^3y = 0$ on any interval.
 - b. Show that the Wronskian is zero only for $x = 0$.
 - c. Joe claims that we have a contradiction to one of our propositions.
 - i. Which proposition is Joe referring to?
 - ii. Is Joe right?

- 3) It is easily verified that the functions $y_1(x) = xe^x$, $y_2(x) = e^{-x}$, are solutions of the equation $y'' - \frac{2}{1+2x}y' - \frac{2x+3}{1+2x}y = 0$ on the interval $(-\frac{1}{2}, \infty)$. Are these functions l.i. on the interval?

- 4) We're given two functions $y_1 = x^3$, $y_2 = |x^3|$, on the interval $I = [-4, 4]$.
 - a. Compute the Wronskian of the functions on I .
 - b. Are the functions 1,2 on I ?
 - c. Could the functions be two solutions of an ODE $y'' + p(x)y' + q(x)y = 0$, with continuous coefficients?
 - d. Note that our functions are solutions of the ODE $xy'' - 2y' = 0$ on I . Does that contradict the result of part c?

5) Answer the following questions:

- a. Let $y_1(x)$, $y_2(x)$ be functions, which are twice continuously differentiable on an interval I and such that their Wronskian is nonzero on I .

Prove that there exists an ODE $y'' + p(x)y' + q(x)y = 0$ with continuous coefficients on I , such that $y_1(x)$, $y_2(x)$ are two of its solutions.

- b. Find an equation $y'' + p(x)y' + q(x)y = 0$ with continuous coefficients on $x > 0$, such that $y_1(x) = x^2$, $y_2(x) = x^4$ are two of its solutions.

Answer Key

1) $\cos x = \sin x$

- 2) a. $W(x) = -2x$ b. This was already shown at a. c. Solution in the recording.

3) Solution in the recording.

- 4) a. $W = 0$ always. b. Solution in the recording. c. No d. No contradiction.

- 5) a. Solution in the recording. b. $y'' - \frac{5}{x}y' + \frac{8}{x^2}y = 0$, ($x > 0$)

Sturm-Liouville Problems

Questions

Solve the following equations:

$$1) \begin{cases} y'' + \lambda y = 0, & 0 \leq x \leq 1 \\ y'(0) = 0 \\ y'(1) = 0 \end{cases}$$

$$2) \begin{cases} y'' + \lambda y = 0, & 0 \leq x \leq 1 \\ y(0) = 0 \\ y(1) + y'(1) = 0 \end{cases}$$

$$3) \begin{cases} y'' + \lambda y = 0, & 0 \leq x \leq 1 \\ y(0) + y'(0) = 0 \\ y(1) = 0 \end{cases}$$

$$4) \begin{cases} y'' + \lambda y = 0, & 0 \leq x \leq l \\ y(0) = 0 \\ y'(l) = 0 \end{cases}$$

$$5) \begin{cases} y'' + \lambda y = 0, & 0 \leq x \leq \pi \\ y'(0) = 0 \\ y(\pi) = 0 \end{cases}$$

$$6) \begin{cases} y'' - 2y' + (1 + \lambda)y = 0, & 0 \leq x \leq 1 \\ y(0) = 0 \\ y(1) = 0 \end{cases}$$

Answer Key

1) $y = 0$

2) $y = 0$

3) $y = 0$

4) $y = 0$

5) $y = 0$

6) $y = 0$