proprep

Workbook

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Classifying Linear 2nd Order PDEs

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Questions

4) Classify the equation $2x^2u_{xx} + 4xyu_{xy} + 2y^2u_{yy} - 2yu_x + ye^{\frac{y}{x}} = 0 \ \ (x, y > 0)$

and bring it to canonical form.

5) Given the equation $2u_{xx} - 2(1+y^2)^2u_{yy} - 4y(1+y^2)u_y = 0$.

a) Show that the equation is hyperbolic.
b) Find its canonical form.
c) Find the general solution to the equation.

6) Given the equation $2u_{xx} + 4u_{xy} - 6u_{yy} = -4(y-3x)^2 + 2\sin(2x+2y)$.

a) Show that the equation is hyperbolic.
b) Find its canonical form.
c) Find the general solution to the equation.

7) Given the equation $u_{xx} + 6u_{xy} + 9u_{yy} = e^x \sin(y-3x)$.

a) Show that the equation is parabolic.
b) Find its canonical form.
c) Find the general solution to the equation.

8) Given the equation $u_{xx} + 2u_{xy} - 3u_{yy} = 0$.

a) Show that the equation is hyperbolic.
b) Find its canonical form.
c) Find the general solution to the equation.

9) Given the equation $u_{xx} + 2u_{xy} + u_{yy} + u_x + u_y = 0$.

a) Show that the equation is parabolic.

b) Find its canonical form.

c) Find the general solution to the equation.

10) Given the equation $u_{xx} - y^2 u_{yy} - y u_y = 0$ on the domain $y > 0$.

a) Show that the equation is hyperbolic.

b) Find its canonical form.

c) Find the general solution to the equation.

d) Find a solution with satisfies $u(0,y) = 2y^2$, $u_x(0,y) = 0$.

11) Given the equation $u_{xx} + u_{xy} - 6u_{yy} + u_x - u = 0$.

a) Show that the equation is hyperbolic.

b) Find its canonical form: $w_{\xi\eta} + \ldots = 0$ (first canonical form).

2^{nd} form: $w_{\xi\xi} - w_{\eta\eta} = 0$

c) Use the change of variables $X = \dfrac{\xi + \eta}{2}, T = \dfrac{\xi - \eta}{2}$

to get an equation of the form $v_{TT} - v_{XX} + \ldots = 0$ (second canonical form).

d) Use the substitution $v(X,T) = z(X,T)e^{\gamma X + \delta T}$ to get an equation

of the form $z_{TT} - z_{XX} = c \cdot z$.

12) Given the equation $u_{\xi\eta} = au_{\xi} + bu_{\eta}$ where $a,b \in \mathbb{R}$ are given constants.

a) Show that the change of variables $x = \dfrac{\xi+\eta}{2}$, $t = \dfrac{\xi-\eta}{2}$

leads to an equation of the form $v_{xx} - v_{tt} + \ldots = 0$.

b) Use a substitution $v(x,t) = z(x,t)e^{\gamma x+\delta t}$ to get an equation

of the form $z_{xx} - z_{tt} = \text{const} \cdot z$.

13) Given the equation $u_{xx} + u_{yy} + au_{x} + bu_{y} + cu = 0$ where $a,b,c \in \mathbb{R}$ are constants.

Use a substitution $u(x,y) = z(x,t)e^{\gamma x+\delta y}$ to get an equation of the form

$z_{xx} - z_{tt} = \text{const} \cdot z$

Answer Key

4) $w_{\eta\eta} + 2\dfrac{\xi^2}{\eta^2} w_\xi + \dfrac{1}{\eta} e^\xi = 0$

5) a) Show b) $w_{\xi\eta} = 0$ c) $u(x,y) = F(\arctan y + x) + G(\arctan y - x)$

6) a) Show b) $w_{\xi\eta} = \dfrac{1}{8}\xi^2 - \dfrac{\sin(2\eta)}{16}$

c)

$$u(x,y) = \frac{1}{24}(y-3x)^3(y+x) + \frac{\cos(2x+2y)}{32}(y-3x) + F(y-3x) + G(y+x)$$

7) a) Show b) $w_{\eta\eta} = e^\eta \sin\xi$

c) $u(x,y) = e^x \sin(y-3x) + f(y-3x)\cdot x + g(y-3x)$

8) a) Show b) $w_{\xi\eta} = 0$ c) $u(x,y) = F(y-3x) + G(y+x)$

9) a) Show b) $w_{\eta\eta} + w_\eta = 0$ c) $u(x,y) = -e^{-x} f(y-x) + g(y-x)$

10) a) Show b) $w_{\xi\eta} = 0$ c) $u(x,y) = F(ye^x) + G(ye^{-x})$

d) $u(x,y) = y^2 e^{2x} + y^2 e^{-2x}$

11) a) Show b) $w_{\xi\eta} = -\dfrac{1}{25}(3w_\xi - 2w_\eta + w)$

c) $v_{TT} - v_{XX} = \dfrac{2}{25}(v_X + 5v_T + 2v)$ d) $z_{TT} - z_{XX} = -\dfrac{124}{625}\cdot z$

12) a) $v_{xx} - v_{tt} = 2(a+b)v_x + 2(a-b)v_t$

b) $z_{xx} - z_{tt} = (4ab)z$

13)

$$\gamma = -\frac{a}{2},\ \delta = -\frac{b}{2}$$

$$z_{xx} + z_{yy} = \left(\frac{a^2}{4} + \frac{b^2}{4} - c\right)z$$