proprep

Workbook

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The Heat Equation

Infinite Interval, Poisson Formula

Questions

1) Given the following IVP with the Heat equation.

$$\begin{cases} u_t = a^2 u_{xx} & -\infty < x < \infty, \quad t \ge 0 \\ u(x,0) = \begin{cases} T_1 & x \ge 0 \\ T_2 & x < 0 \end{cases} \end{cases}$$

where T_1, T_2 are real constants (temperatures).

Solve the IVP for $u(x,t)$ and compute $\lim_{t\to\infty} u(x,t)$.

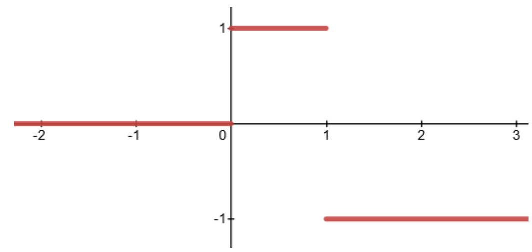
You may use the error function $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$.

2) Solve the following IVP with Heat equation.

$$\begin{cases} u_t = a^2 u_{xx} & -\infty < x < \infty, \quad t \ge 0 \\ u(x,0) = e^x \end{cases}$$

3) Solve the following IVP with Heat equation:

$$\begin{cases} u_t = u_{xx} & -\infty < x < \infty, \quad t \ge 0 \\ u(x,0) = \begin{cases} 0 & x < 0 \\ 1 & 0 < x < 1 \\ -1 & x > 1 \end{cases} \end{cases}$$



You may use the error function $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$

4) Solve the following IVP with Heat equation:

$$\begin{cases} u_t = u_{xx} & -\infty < x < \infty, \quad t \ge 0 \\ u(x,0) = e^{-x^2} \end{cases}$$

5) Solve the following IVP: heat equation $u_t = u_{xx}$, $-\infty < x < \infty$, $t \ge 0$ with initial condition

a) $u(x,0) = 2$

b) $u(x,0) = x^2$ Hint: define $q = u_t$

6) Given the following IVP.

$$\begin{cases} u_t = u_{xx} - u & -\infty < x < \infty, \quad t \ge 0 \\ u(x,0) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases} \end{cases}$$

a) Find $u(x,t)$. Suggestion: define $u(x,t) = e^{\alpha x + \beta v} v(x,t)$.

b) Compute the integral $\int_{-\infty}^{\infty} u_x^2(x,t)\,dx$.

7) Solve the following IVP:

$$\begin{cases} \overset{\text{heat equation}}{u_t = u_{xx}} \ , \ -\infty < x < \infty \ , \ t \geq 0 \\ u(x,0) = \cos x \end{cases}$$

Hint: define $w(x,t) = u_t(x,t)$

Answer Key

1) $u(x,t)=\frac{T_1+T_2}{2}+\frac{T_1-T_2}{2}\operatorname{erf}\left(\frac{x}{2a\sqrt{t}}\right)$

$\lim_{t\to\infty}u(x,t)=\frac{T_1+T_2}{2}$

2) $u(x,t)=e^{x+a^2t}$

3) $u(x,t)=\frac{1}{2}\operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right)+\operatorname{erf}\left(\frac{1-x}{2\sqrt{t}}\right)-\frac{1}{2}$

4) $u(x,t)=\frac{1}{\sqrt{4t+1}}e^{-\frac{x^2}{4t+1}}$

5) a) $u(x,t)=2$

b) $u(x,t)=2t+x^2$

6) a) $u(x,t)=\frac{1}{2}e^{-t}\left[\operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right)+1\right]$

b) $\int_{-\infty}^{\infty}u_x^2(x,t)dx=2^{-\frac{3}{2}}\frac{1}{\sqrt{\pi t}}e^{-2t}$

7) $u(x,t)=e^{-t}\cos x$

Finite Interval, Separation of Variables

Questions

1) Solve the following IBVP.

$$\begin{cases} u_t = u_{xx} & 0 \le x \le \pi,\ t \ge 0 \\ u(x,0) = \sin x \\ u(0,t) = u(\pi,t) = 0 \end{cases}$$

$$a = 1,\ L = \pi,\ f(x) = \sin x$$

2) Solve the following IBVP.

$$\begin{cases} u_t = u_{xx} & 0 \le x \le \pi,\ t \ge 0 \\ u(x,0) = \frac{1}{2} + 3\sin^2 x \\ u_x(0,t) = u_x(\pi,t) = 0 \end{cases}$$

$$a = 1,\ L = \pi,\ f(x) = \frac{1}{2} + 3\sin^2 x$$

3) Solve the following IBVP.

$$\begin{cases} u_t = 16u_{xx} & 0 \le x \le 3,\ t \ge 0 \\ u(x,0) = x \\ u_x(0,t) = u_x(3,t) = 0 \end{cases}$$

$$a = 4,\ L = 3,\ f(x) = x$$

4) Solve the following IBVP.

$$\begin{cases} u_t = 9u_{xx} & 0 \le x \le 2,\ t \ge 0 \\ u(x,0) = 6 + 4\cos\left(\frac{3\pi}{2}x\right) \\ u_x(0,t) = u_x(2,t) = 0 \end{cases}$$

5) Solve the following IBVP.

$$\begin{cases} u_t = 9u_{xx} & 0 \le x \le 2,\ t \ge 0 \\ u(x,0) = x \\ u_x(0,t) = u_x(2,t) = 0 \end{cases}$$

$$a = 3,\ L = 2,\ f(x) = x$$

6) Solve the following IBVP.

$$\begin{cases} u_t = u_{xx} & 0 \le x \le \pi,\ t \ge 0 \\ u(x,0) = x \\ u(0,t) = u(\pi,t) = 0 \end{cases}$$

7) Solve the following IBVP.

$$\begin{cases} u_t = 4u_{xx} + \frac{1}{2}\sin(2x) & 0 \le x \le \pi,\ t \ge 0 \\ u(x,0) = \sin(x)(1-\cos(3x)) \\ u(0,t) = u(\pi,t) = 0 \end{cases}$$

Hint: Find a correction function w, such that $u(x,t) = v(x,t) + w(x)$,

to get a homogeneous

heat equation in v.

Answer Key

1) $u(x,t)=e^{-t}\sin x$

2) $u(x,t)=2-\frac{3}{2}e^{-4t}\cos(2x)$

3) $u(x,t)=\frac{3}{2}+\sum_{k=1}^{\infty}\frac{-12}{(2k-1)^2\pi^2}e^{-\left(\frac{4(2k-1)\pi}{3}\right)^2 t}\cos\left(\frac{(2k-1)\pi}{3}x\right)$

4) $u(x,t)=6+4e^{-\frac{81\pi^2}{4}t}\cos\left(\frac{3\pi}{2}x\right)$

5) $u(x,t)=1+\sum_{k=1}^{\infty}\frac{-8}{(2k-1)^2\pi^2}e^{-\left(\frac{3(2k-1)\pi}{2}\right)^2 t}\cos\left(\frac{(2k-1)\pi}{2}x\right)$

6) $u(x,t)=-2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin(nx)e^{-n^2t}$

7) $u(x,t)=e^{-4t}\sin(x)+\left(\frac{15e^{-16t}+1}{32}\right)\sin(2x)-\frac{1}{2}e^{-64t}\sin(4x)$

Duhamel's Principle

Questions

1) Solve the following IBVP, using Duhamel's Principle:

$$\begin{cases} u_t = a^2 u_{xx} + 6\sin\left(\dfrac{2\pi x}{L}\right) & 0 \le x \le L, \quad t \ge 0 \\ u(x,0) = 0 & \\ u(0,t) = u(L,t) = 0 & \end{cases}$$

2) Solve the following IVP, using Duhamel's Principle:

$$\begin{cases} u_t = a^2 u_{xx} + x^2 t & -\infty < x < \infty, \quad t \ge 0 \\ u(x,0) = 0 & \end{cases}$$

3) Solve the following IVP, using Duhamel's Principle:

$$\begin{cases} u_t = a^2 u_{xx} + 2x\sin t & -\infty < x < \infty, \quad t \ge 0 \\ u(x,0) = 0 & \end{cases}$$

4) Solve the following IBVP, using Duhamel's Principle:

$$\begin{cases} u_t = u_{xx} + e^t \cos(\pi x) & 0 \le x \le 1, \quad t \ge 0 \\ u(x,0) = 0 & \\ u_x(0,t) = u_x(1,t) = 0 & \end{cases}$$

5) Solve the following IBVP, using Duhamel's Principle:

$$\begin{cases} u_t = a^2 u_{xx} + e^{-\left(\frac{\pi a}{L}\right)^2 t} \sin\left(\dfrac{\pi x}{L}\right) \quad , \qquad 0 \le x \le L, \quad t \ge 0 \\ u(x,0) = 0 \\ u(0,t) = u(L,t) = 0 \end{cases}$$

Answer Key

1) $u(x,t)=6\left(\frac{L}{2\pi a}\right)^2 e^{-\left(\frac{2\pi a}{L}\right)^2 t}\sin\left(\frac{2\pi}{L}x\right)\left[1-e^{\left(\frac{2\pi a}{L}\right)^2 t}\right]$

2) $u(x,t)=\frac{1}{2}x^2t^2+\frac{1}{3}a^2t^3$

3) $u(x,t)=2x(1-\cos t)$

4) $u(x,t)=\frac{e^t-e^{-\pi^2 t}}{1+\pi^2}\cos(\pi x)$

5) $u(x,t)=te^{-\left(\frac{\pi a}{L}\right)^2 t}\sin\left(\frac{\pi x}{L}\right)$

The Maximum and Minimum Principle

Questions

1) Given the following two IBVPs on the domain $0 \le x \le 1,\ t \ge 0$:

$$\begin{cases} u_t = u_{xx} + e^t \cos x \\ u(x,0) = x^2 \\ u(0,t) = u(1,t) = t^2 \end{cases} \qquad \begin{cases} v_t = u_{xx} + e^t \cos x \\ v(x,0) = x \\ v(0,t) = v(1,t) = t \end{cases}$$

Determine which expression is greater: $u\left(\frac{1}{2},\frac{1}{2}\right)$ or $v\left(\frac{1}{2},\frac{1}{2}\right)$

2) Given the following two IBVPs on the domain $0 \le x \le 1,\ t \ge 0$:

$$\begin{cases} u_t = u_{xx} + \sin t \cos x \\ u(x,0) = \sin(\pi x) \\ u(0,t) = u(1,t) = 0 \end{cases} \qquad \begin{cases} v_t = u_{xx} + \sin t \cos x \\ v(x,0) = 0 \\ v(0,t) = v(1,t) = -\sin^2(\pi t) \end{cases}$$

Determine which expression is greater: $u\left(\frac{1}{2},1\right)$ or $v\left(\frac{1}{2},1\right)$

3) Given the following IBVP on the domain $0 \le x \le 1,\ t \ge 0$:

$$\begin{cases} u_t = u_{xx} - u \\ u(x,0) = x(x-1) \\ u(0,t) = u(1,t) = 0 \end{cases}$$

Prove that $-\frac{1}{4e} < u\left(\frac{1}{2},1\right) < 0$.

Hint: define $h(x,t) = u(x,t)e^{\delta t}$ for a suitable constant δ.

4) Given the following IBVP with nonhomogeneous heat equation:

$$\begin{cases} u_t = u_{xx} - xe^{-t}(1-x), & 0 \le x \le 1,\ t \ge 0 \\ u(x,0) = \sin(\pi x) \\ u(0,t) = u(1,t) = 0 \end{cases}$$

Prove that $u(x,t) < 1$ for $0 < x < 1,\ t > 0$.

5) Given the following IBVP with nonhomogeneous heat equation:

$$\begin{cases} u_t = u_{xx} - \sin(\pi x), & 0 \le x \le 1,\ t \ge 0 \\ u(x,0) = x \\ u(0,t) = \sin(\pi t),\ \ u(1,t) = \cos(\pi t) \end{cases}$$

Prove that $u\left(\frac{1}{2}, \pi\right) < 1$.

6) Given the following IBVP (with nonhomogeneous heat equation):

$$\begin{cases} u_t = u_{xx} + F(x,t), & 0 \le x \le L,\ t \ge 0 \\ u(x,0) = f(x) \\ u(0,t) = g(t),\ \ u(L,t) = h(t) \end{cases}$$

Prove that the solution (if it exists) is unique.

Answer Key

1) $u\left(\frac{1}{2},\frac{1}{2}\right) < v\left(\frac{1}{2},\frac{1}{2}\right)$

2) $u\left(\frac{1}{2},1\right) > v\left(\frac{1}{2},1\right)$

3) Prove

4) Prove

5) Prove

6) Prove

Asymptotic Behaviour of u(x,t) Infinite Interval

Questions

1) Given the following IBVP:

$$\begin{cases} u_t = a^2 u_{xx} & -\infty < x < \infty, \;\; t \geq 0 \\ u(x,0) = x^2 e^{-|x|} \end{cases}$$

Compute $\lim_{t\to\infty} t^{\frac{1}{3}} u(1,t)$

2) Given the following IBVP:

$$\begin{cases} u_t = a^2 u_{xx} & -\infty < x < \infty, \;\; t \geq 0 \\ u(x,0) = 2xe^{-x^2} \end{cases}$$

Compute $\lim_{t\to\infty} \sqrt{t} \cdot u(x,t)$ for all real x.

3) Given that u is a solution of the following IBVP:

$$\begin{cases} u_t = a^2 u_{xx} & -\infty < x < \infty, \;\; t \geq 0 \\ u(x,0) = e^{-|x|} \end{cases}$$

Compute $\lim_{t\to\infty} \sqrt{t} \cdot u(x,t)$ for all real x.

4) Given the following IBVP:

$$\begin{cases} u_t = a^2 u_{xx} & -\infty < x < \infty, \;\; t \geq 0 \\ u(x,0) = 1 + e^{-|x|} = h(x) \end{cases}$$

Compute $\lim_{t\to\infty} u(x,t)$ for all real x.

Answer Key

1) $\lim_{t\to\infty} t^{\frac{1}{3}} u(1,t) = 0$

2) $\lim_{t\to\infty} \sqrt{t}\cdot u(x,t) = 0$

3) $\lim_{t\to\infty} \sqrt{t}\cdot u(x,t) = \frac{1}{a\sqrt{\pi}}$

4) $\lim_{t\to\infty} u(x,t) = 1$

Asymptotic Behaviour of u(x,t) Finite Interval

Questions

1) Given the following IBVP:

$$\begin{cases} u_t = u_{xx} + e^{-t} & 0 \le x \le 1,\ \ t \ge 0 \\ u(x,0) = 1 \\ u_x(0,t) = 1 \qquad u_x(1,t) = \dfrac{t^2}{t^2+1} \end{cases}$$

Compute $U(x) = \lim_{t\to\infty} u(x,t)$ [without computing $u(x,t)$]

2) Given the following IBVP:

$$\begin{cases} u_t = u_{xx} + x & -1 \le x \le 1,\ \ t \ge 0 \\ u(x,0) = 0 \\ u_x(-1,t) = u_x(1,t) = 0 \end{cases}$$

Compute $U(x) = \lim_{t\to\infty} u(x,t)$ [without computing $u(x,t)$]

3) Given the following IBVP:

$$\begin{cases} u_t = u_{xx} + \dfrac{1}{(t+1)^{\alpha}} & 0 \le x \le 1,\ \ t \ge 0,\ \ (\alpha > 1) \\ u(x,0) = x \\ u_x(0,t) = u_x(1,t) = 0 \end{cases}$$

Compute $U(x) = \lim_{t\to\infty} u(x,t)$ [without computing $u(x,t)$]

Answer Key

1) $U(x) = x + \frac{3-\pi}{2}$

2) $U(x) = -\frac{x^3}{6} + \frac{1}{2}x$

3) $U(x) = \frac{1}{2} - \frac{1}{1-\alpha}$