proprep

Workbook

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The Wave Equation

Infinite Interval

Questions

1) Solve the following problem:

$$\begin{cases} u_{tt} = u_{xx} & -\infty < x < \infty \,,\, t \geq 0 \\ u(x,0) = 0 & \\ u_t(x,0) = |x| & \end{cases}$$

2) Solve the following problem:

$$\begin{cases} u_{tt} = u_{xx} & (-\infty < x < \infty \,,\, t \geq 0) \\ u(x,0) = 0, \quad u_t(x,0) = \dfrac{|x|}{1+x^2} & \end{cases}$$

3) Solve the following problem:

$$u_{tt} = u_{xx} + x^2 + t^2 \qquad -\infty < x < \infty \,,\, t \geq 0$$

$$u(x,0) = x, \; u_t(x,0) = x$$

4) Solve the following problem:

$$\begin{cases} u_{tt} = a^2 u_{xx} & -\infty < x < \infty \,,\, t \geq 0 \\ u(x,0) = e^{-x} & \\ u_t(x,0) = x & \end{cases}$$

5) Solve the following problem:

$$\begin{cases} u_{tt} = a^2 u_{xx} & -\infty < x < \infty \,,\, t \geq 0 \\ u(x,0) = \sin x \\ u_t(x,0) = \cos x \end{cases}$$

6) Given that $u(x,t)$ is a solution of the following problem:

$$u_{tt} = u_{xx} \qquad -\infty < x < \infty \,,\, t \geq 0$$

$$u(x,0) = \begin{cases} 0 & x < 2 \\ 4 & x > 2 \end{cases}, \qquad u_t(x,0) = 0$$

Compute $\int_{-\infty}^{4} u(x,1)dx$

7) Given that $u(x,t)$ is a solution of the following problem:

$$u_{tt} = u_{xx} \qquad -\infty < x < \infty \,,\, t \geq 0$$

$$u(x,0) = 0, \qquad u_t(x,0) = e^{-|x|}$$

a) Find $u(x,t)$

b) Compute $\int_{0}^{10} u_x(x,5)dx$

8) Given that $u(x,t)$ is a solution of the following problem:

$$u_{tt} = u_{xx} \qquad -\infty < x < \infty \; , \; t \geq 0$$

$$u(x,0) = \begin{cases} 1 - x^2 & |x| < 1 \\ 0 & \text{else} \end{cases} , \qquad u_t(x,0) = 0$$

a) Find $u(x,t)$

b) Compute $\int_0^2 u(x,1)dx$

9) Solve the following problem:

$$u_{tt} = u_{xx} + t\cos x \qquad -\infty < x < \infty \; , \; t \geq 0$$

$$u(x,0) = 0 \qquad u_t(x,0) = \cos x$$

Note: you may use the identity $\frac{\sin(a+b) - \sin(a-b)}{2} = \cos a \sin b$

10) Solve the following problem:

$$u_{tt} = u_{xx} - 2e^x \sin t \qquad -\infty < x < \infty \; , \; t \geq 0$$

$$u(x,0) = \sin x \qquad u_t(x,0) = 1$$

11) a) Show that the general solution to the equation $u_{tt} = u_{xx}$ is $u(x,t) = f(x+t) + g(x-t)$.

b) Given continuously differentiable functions $a(t), b(t)$ satisfying $a(0) = b(0) = 0$.

Solve the problem $u_{tt} = u_{xx}$ with conditions $u(t,t) = a(t), u(-t,t) = b(t)$.

12) Solve the following problem:

$$u_{tt} = u_{xx} - 2u_t - u \qquad -\infty < x < \infty \,,\, t \ge 0$$

$$u_{tt} = u_{xx} + F(x,t)$$

$$u(x,0) = \sin x \qquad u_t(x,0) = 0$$

Hint: define $v(x,t) = u(x,t)e^{-(\alpha x + \beta t)}$ for suitable constants α, β .

13) Solve the following problem:

$$u_{tt} = u_{xx} \ , \ -\infty < x < \infty \,,\, t \ge \begin{cases} -\frac{3}{2}x & x < 0 \\ 0 & x > 0 \end{cases}$$

$$u\left(x, -\frac{3}{2}x\right) = 0 \qquad (x < 0)$$

$$u(x,0) = u_t(x,0) = \sin x \qquad (x \ge 0)$$

Hint: the general solution of $u_{tt} = a^2 u_{xx}$ is of the form $u(x,t) = f(x - at) + g(x + at)$.

14) Solve the following problem:

$$u_{tt} = \alpha^2 u_{xx} \quad (\alpha > 0) \qquad -\infty < x < \infty \,,\, t \ge 0$$

$$u(x,0) = 0 \qquad u_t(x,0) = |x|$$

15) Solve the following problem:

$$u_{tt} = u_{xx} \qquad -\infty < x < \infty \,,\, t \ge 0$$

$$u(x,0) = \begin{cases} 1 - x^2 & |x| \le 1 \\ 0 & \text{else} \end{cases} \qquad u_t(x,0) = 0$$

Answer Key

1) $$u(x,t)=\begin{cases} xt & x>t \\ \frac{1}{2}(x^2+t^2) & -t<x<t \\ -xt & x<-t \end{cases}$$

2) $$u(x,t)=\begin{cases} \frac{1}{4}\left[\ln(1+[x+t]^2)-\ln(1+[x-t]^2)\right] & x>t \\ \frac{1}{4}\left[\ln(1+[x-t]^2)+\ln(1+[x+t]^2)\right] & -t<x<t \\ \frac{1}{4}\left[\ln(1+[x-t]^2)-\ln(1+[x+t]^2)\right] & x<-t \end{cases}$$

3) $$u(x,t)=x+xt+\frac{1}{6}t^4+\frac{1}{2}x^2t^2$$

4) $$u(x,t)=e^{-x}\cosh(at)+xt$$

5) $$u(x,t)=\frac{(a+1)\sin(x+at)+(a-1)\sin(x-at)}{2a}$$

6) 8

7) a) $u(x,t)=\begin{cases}\dfrac{e^{x+t}-e^{x-t}}{2} & x<-t\\ \dfrac{2-e^{x-t}-e^{-x-t}}{2} & -t<x<t\\ \dfrac{e^{-x+t}-e^{-x-t}}{2} & x>t\end{cases}$

b) $\dfrac{-2+3e^{-5}-e^{-15}}{2}$

8) a) $u(x,1)=\dfrac{1}{2}\begin{cases}1-(x+1)^2 & -2<x<0\\ 1-(x-1)^2 & 0<x<2\\ 0 & \text{else}\end{cases}$

b) $\dfrac{2}{3}$

9) $u(x,t)=t\cos x$

10) $u(x,t)=\sin(x)\cos(t)+t+e^{-x}\sin t-e^{-x}\sinh t$

11) a) Show

b) $u(x,t)=a\left(\dfrac{t+x}{2}\right)+b\left(\dfrac{t-x}{2}\right)$

12) $u(x,t)=\sin(x)[\cos(t)+2\sin(t)]e^{-t}$

13) $$u(x,t)=\begin{cases}\frac{1}{2}\sin\left(\frac{x-t}{5}\right)+\frac{1}{2}\cos\left(\frac{x-t}{5}\right)+\frac{1}{2}\sin(x+t)-\frac{1}{2}\cos(x+t) & -\frac{2}{3}t\le x<t\\ \frac{1}{2}\left[\sin(x-t)+\cos(x-t)\right]+\frac{1}{2}\left[\sin(x+t)-\cos(x+t)\right] & 0\le t\le x\end{cases}$$

14) $$u(x,t)=\begin{cases}-xt & x<-\alpha t\\ \frac{x^2+\alpha^2t^2}{2\alpha} & -\alpha t\le x\le \alpha t\\ xt & x>\alpha t\end{cases}$$

15) $$u(x,t)=\begin{cases}\frac{1-(x+t)^2}{2} & -1-t\le x\le 1-t\\ 0 & \text{else}\end{cases}+\begin{cases}\frac{1-(x-t)^2}{2} & -1+t\le x\le 1+t\\ 0 & \text{else}\end{cases}$$

Semi-Infinite Interval

Questions

1) Given the following IBVP, find $u\left(x,\frac{1}{2}\right)$.

$$\begin{cases} u_{tt} = u_{xx} & 0 \le x < \infty \,,\, t \ge 0 \\ u(x,0) = f(x) = 0 & \\ u_t(x,0) = g(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{else} \end{cases} & \\ u(0,t) = 0 \text{ (Dirichlet BC)} & \end{cases}$$

2) Solve the following IBVP (initial/boundary value problem):

$$\begin{cases} u_{tt} = u_{xx} & 0 \le x < \infty \,,\, t \ge 0 \\ u(x,0) = f(x) = x & \\ u_t(x,0) = g(x) = \sin x & \\ u(0,t) = 0 \text{ (Dirichlet BC)} & \end{cases}$$

3) Solve the following IBVP (initial/boundary value problem):

$$\begin{cases} u_{tt} = u_{xx} & 0 \le x < \infty \,,\, t \ge 0 \\ u(x,0) = f(x) = x^2 & \\ u_t(x,0) = g(x) = \sin x & \\ u(0,t) = 0 & \end{cases}$$

4) Solve the following IBVP (initial/boundary value problem):

$$\begin{cases} u_{tt} = u_{xx} & 0 \le x < \infty \ , \ t \ge 0 \\ u(x,0) = f(x) = x^2 \\ u_t(x,0) = g(x) = \sin x \\ u_x(0,t) = 0 \end{cases}$$

5) Solve the following IBVP:

$$\begin{cases} u_{tt} = u_{xx} & 0 \le x < \infty \ , \ t \ge 0 \\ u(x,0) = f(x) = 2x^2 - x \\ u_t(x,0) = g(x) = 4x + 1 \\ u(0,t) = 0 \end{cases}$$

6) Given the following two IBVPs:

$$\begin{cases} u_{tt} = u_{xx} & 0 \le x < \infty \ , \ t \ge 0 \\ u(x,0) = x^3 \\ u_t(x,0) = 0 \\ u(0,t) = 0 \end{cases} \qquad \begin{cases} v_{tt} = v_{xx} & 0 \le x < \infty \ , \ t \ge 0 \\ v(x,0) = x^3 \\ v_t(x,0) = 0 \\ v_x(0,t) = 0 \end{cases}$$

Compute the limit $\lim_{t \to \infty} \frac{u(1,t)}{v(1,t)}$

7) Solve the following wave equation:

$$\begin{cases} u_{tt} = u_{xx} + t & 0 \le x < \infty \,,\ t \ge 0 \\ u(x,0) = x \\ u_t(x,0) = 0 \\ u(0,t) = \frac{1}{6}t^3 \end{cases}$$

8) Given that $u(x,t)$ is a solution of the following wave equation:

$$\begin{cases} u_{tt} = u_{xx} + 2u_x + u & 0 \le x < \infty \,,\ t \ge 0 \\ u(x,0) = e^{-x} \\ u_t(x,0) = e^{-2x} \\ u(0,t) = 0 \end{cases} \qquad u_{tt} = u_{xx} + F(x,t)$$

Compute $\lim_{x\to\infty}[e^x \cdot u(x,2)]$. Hint: define $v(x,t) = u(x,t)\cdot e^{-(\alpha x+\beta t)}$

9) Solve the following wave equation:

$$\begin{cases} u_{tt} = u_{xx} & 0 \le x < \infty \,,\ t \ge 0 \\ u(x,0) = \sin^2 x \\ u_t(x,0) = \sin x + 1 \\ u(0,t) = t \end{cases}$$

10) Given the following IBVP (initial/boundary value problem):

$$\begin{cases} u_{tt} = \alpha^2 u_{xx} \ (\alpha > 0) & 0 \le x < \infty \ , t \ge 0 \\ u(x,0) = f(x) = x^2 \\ u_t(x,0) = g(x) = \sin^2 x \\ u(0,t) = 0 \end{cases}$$

Answer Key

1) $u\left(x,\frac{1}{2}\right)=\begin{cases}0 & x>\frac{3}{2}\\ \frac{1}{2}\left(\frac{3}{2}-x\right) & \frac{1}{2}<x<\frac{3}{2}\\ x & 0<x<\frac{1}{2}\end{cases}$

2) $u(x,t)=x+\sin(x)\sin(t)$

3) $u(x,t)=\begin{cases}x^2+t^2+\sin(x)\sin(t) & x>t\\ 2xt+\sin(x)\sin(t) & 0<x<t\end{cases}$

4) $\begin{cases}x^2+t^2+\sin(x)\sin(t) & x\geq t\\ x^2+t^2+1-\cos(x)\cos(t) & 0\leq x<t\end{cases}$

5) $u(x,t)=\begin{cases}2(x+t)^2-x+t & x\geq t\\ 8xt & 0\leq x<t\end{cases}$

6) 0

7) $u(x,t)=x+\frac{1}{6}t^3$

8) 1

9) $$u(x,t)=\begin{cases}\dfrac{\sin^2(x+t)+\sin^2(x-t)}{2}+\sin(x)\sin(t)+t & x>t\\[2ex] \dfrac{\sin^2(x+t)-\sin^2(x-t)}{2}+\sin(x)\sin(t)+t & 0<x<t\end{cases}$$

10) $$u(x,t)=\begin{cases}2\alpha xt+\dfrac{x}{2\alpha}-\dfrac{\sin(2x)\cos(2\alpha t)}{4\alpha} & 0\le x<\alpha t\\[2ex] x^2+\alpha^2t^2+\dfrac{t}{2}-\dfrac{\cos(2x)\sin(2\alpha t)}{4\alpha} & x\ge \alpha t\end{cases}$$

Finite, Interval, Homogeneous Equation

Questions

1) Solve the following Wave Equation using separation of variables.

$$\begin{cases} u_{tt} = u_{xx} & 0 \le x \le 1, \quad t \ge 0 \\ u(x,0) = \frac{1}{2}\sin(\pi x) - 7\sin(5\pi x) \\ u_t(x,0) = 0 \\ u(0,t) = u(1,t) = 0 \end{cases}$$

2) Solve the following Wave Equation using separation of variables.

$$\begin{cases} u_{tt} = u_{xx} & 0 \le x \le 1, \quad t \ge 0 \\ u(x,0) = 1 \\ u_t(x,0) = 0 \\ u(0,t) = u(1,t) = 0 & \text{(fixed endpoints)} \end{cases}$$

3) Solve the following Wave Equation using separation of variables.

$$\begin{cases} u_{tt} = 4u_{xx} & 0 \le x \le 1, \quad t \ge 0 \\ u(x,0) = -2x - 1 \\ u_t(x,0) = 0 \\ u(0,t) = u(1,t) = 0 \end{cases}$$

4) Solve the following Wave Equation using separation of variables.

$$\begin{cases} u_{tt} = a^2 u_{xx} & 0 \le x \le L, \quad t \ge 0 \\ u(x,0) = \cos\left(\dfrac{\pi}{2L}x\right) \\ u_t(x,0) = \cos\left(\dfrac{3\pi}{2L}x\right) + \cos\left(\dfrac{5\pi}{2L}x\right) \\ \overset{\text{Neumann}}{u_x(0,t)} = \overset{\text{Dirichlet}}{u(L,t)} = 0 \end{cases}$$

5) Solve the following Wave Equation using separation of variables.

$$\begin{cases} u_{tt} = a^2 u_{xx} & 0 \le x \le L, \quad t \ge 0 \\ u(x,0) = \sin\left(\dfrac{5\pi}{2L}x\right) \\ u_t(x,0) = \sin\left(\dfrac{\pi}{2L}x\right) \\ \overset{\text{Dirichlet}}{u(0,t)} = \overset{\text{Neumann}}{u_x(L,t)} = 0 \end{cases}$$

6) Solve the following Wave Equation:

$$\begin{cases} u_{tt} = u_{xx} & 0 \le x \le 1, \quad t \ge 0 \\ u(x,0) = 1 + \cos(\pi x) + \cos(3\pi x) \\ u_t(x,0) = 1 \\ u_x(0,t) = u_x(1,t) = 0 \end{cases}$$

7) Solve the following Wave Equation:

$$\begin{cases} u_{tt} = u_{xx} & 0 \le x \le 1, \quad t \ge 0 \\ u(x,0) = \cos(\pi x) \\ u_t(x,0) = \cos(2\pi x) \\ u_x(0,t) = u_x(1,t) = 0 \end{cases}$$

Answer Key

1) $u(x,t)=\frac{1}{2}\sin(\pi x)\cos(\pi t)-7\sin(5\pi x)\cos(5\pi t)$

2) $u(x,t)=\sum_{k=1}^{\infty}\frac{-4}{\pi(2k-1)}\sin((2k-1)\pi x)\cos((2k-1)\pi t)$

3) $u(x,t)=\sum_{n=1}^{\infty}2\frac{3(-1)^n-1}{\pi n}\sin(n\pi x)\cos(2n\pi t)$

4) $u(x,t)=\cos\left(\frac{\pi}{2L}x\right)\cos\left(\frac{\pi}{2L}at\right)+\frac{2L}{3\pi a}\cos\left(\frac{3\pi}{2L}x\right)\sin\left(\frac{3\pi}{2L}at\right)+\frac{2L}{5\pi a}\cos\left(\frac{5\pi}{2L}x\right)\sin\left(\frac{5\pi}{2L}at\right)$

5) $u(x,t)=\frac{2L}{\pi a}\sin\left(\frac{\pi}{2L}x\right)\sin\left(\frac{\pi}{2L}at\right)+\sin\left(\frac{5\pi}{2L}x\right)\cos\left(\frac{5\pi}{2L}at\right)$

6) $u(x,t)=1+t+\cos(\pi x)\cos(\pi t)+\cos(3\pi x)\cos(3\pi t)$

7) $u(x,t)=\cos(\pi x)\cos(\pi t)+\frac{1}{2\pi}\cos(2\pi x)\sin(2\pi t)$

Finite, Interval, Nonhomogeneous Equation

Questions

1) Solve the following nonhomogeneous Wave Equation on a finite interval:

$$\begin{cases} u_{tt} = a^2 u_{xx} + (t+1)\sin\left(\dfrac{3\pi x}{L}\right) - \dfrac{9x}{L}\sin(3t) & 0 \le x \le L, \quad t \ge 0 \\ u(x,0) = 0 \qquad u_t(x,0) = \dfrac{3x}{L} \\ u(0,t) = 0 \qquad u(L,t) = \sin(3t) \end{cases}$$

Hint: use a correction function of the form $w(x,t) = A(t) + B(t)x$

2) Solve the following nonhomogeneous Wave Equation on a finite interval:

$$\begin{cases} u_{tt} = a^2 u_{xx} + \sin\dfrac{\pi x}{L} & 0 \le x \le L, \quad t \ge 0 \\ u(x,0) = 0 \qquad u_t(x,0) = 0 \\ u(0,t) = 0 \qquad u(L,t) = 0 \end{cases}$$

3) Solve the following nonhomogeneous Wave Equation on a finite interval:

$$\begin{cases} u_{tt} = a^2 u_{xx} + t\sin\dfrac{2\pi x}{L} & 0 \le x \le L, \quad t \ge 0 \\ u(x,0) = \sin\dfrac{2\pi x}{L} \qquad u_t(x,0) = 0 \\ u(0,t) = 0 \qquad u(L,t) = 0 \end{cases}$$

4) Solve the following nonhomogeneous Wave Equation on a finite interval:

$$\begin{cases} u_{tt} = a^2 u_{xx} + \cos\dfrac{\pi x}{L} & 0 \le x \le L, \quad t \ge 0 \\ u(x,0) = \cos\dfrac{\pi x}{L} & u_t(x,0) = 0 \\ u_x(0,t) = 0 & u_x(L,t) = 0 \end{cases}$$

5) Solve the following nonhomogeneous Wave Equation on a finite interval:

$$\begin{cases} u_{tt} = a^2 u_{xx} + e^t & 0 \le x \le L, \quad t \ge 0 \\ u(x,0) = \cos\dfrac{\pi x}{L} & u_t(x,0) = 0 \\ u_x(0,t) = 0 & u_x(L,t) = 0 \end{cases}$$

Answer Key

1) $u(x,t)=\sin\frac{3\pi x}{L}\left[\left(\frac{L}{3\pi a}\right)^2(t+1)-\left(\frac{L}{3\pi a}\right)^2\cos\left(\frac{3\pi a}{L}t\right)-\left(\frac{L}{3\pi a}\right)^3\sin\left(\frac{3\pi a}{L}t\right)\right]+\frac{1}{L}\sin(3t)x$

2) $u(x,t)=\sin\frac{\pi x}{L}\left[-\left(\frac{L}{\pi a}\right)^2\cos\left(\frac{\pi a}{L}t\right)+\left(\frac{L}{\pi a}\right)^2\right]$

3) $u(x,t)=\sin\frac{\pi x}{L}\cos\left(\frac{\pi a}{L}t\right)+\sin\frac{2\pi x}{L}\left[\left(\frac{L}{2\pi a}\right)^2 t-\left(\frac{L}{2\pi a}\right)^3\sin\left(\frac{2\pi a}{L}t\right)\right]$

4) $u(x,t)=\cos\frac{\pi x}{L}\left\{\left[1-\left(\frac{L}{\pi a}\right)^2\right]\cos\left(\frac{\pi a}{L}t\right)+\left(\frac{L}{\pi a}\right)^2\right\}$

5) $u(x,t)=(e^t-t-1)+\cos\frac{\pi x}{L}\cos\frac{\pi a t}{L}$

Domain of Dependence

Questions

1) Given the following Wave Equation. Compute $u\left(\frac{\pi}{2},\frac{1}{4}\right)$.

$$\begin{cases} u_{tt} = u_{xx} & 0 \le x \le 2, \quad t \ge 0 \\ u(x,0) = x & u_t(x,0) = \cos x \\ u(0,t) = \cos t & u(L,t) = \sin t \end{cases}$$

2) Given the following Wave Equation. Compute $u\left(\frac{1}{2},\frac{1}{4}\right)$.

$$\begin{cases} u_{tt} = u_{xx} & 0 \le x \le 1, \quad t \ge 0 \\ u(x,0) = x(x-1) & u_t(x,0) = 1 \\ u_x(0,t) = 0 & u_x(1,t) = 0 \end{cases}$$

3) Given the following two IBVPs on the domain $\{0 \le x \le 1,\ t \ge 0\}$

$$\begin{cases} u_{tt} = u_{xx} + e^{-t}\cos(\pi x) \\ u(x,0) = \cos^2(\pi x) \\ u_t(x,0) = -\sin^2(\pi x) \\ u_x(0,t) = u_x(1,t) = 0 \end{cases} \qquad \begin{cases} v_{tt} = v_{xx} + e^{-t}\cos(\pi x) \\ v(x,0) = -\sin^2(\pi x) \\ v_t(x,0) = \cos^2(\pi x) \\ v_x(0,t) = v_x(1,t) = 0 \end{cases}$$

Determine which is greater: $u\left(\frac{1}{2},\frac{1}{2}\right)$ or $v\left(\frac{1}{2},\frac{1}{2}\right)$

4) Given the following Wave Equation. Compute $u\left(1,\frac{1}{2}\right)$.

$$\begin{cases} u_{tt} = u_{xx} + t^2 & 0 \le x \le 2, \quad t \ge 0 \\ u(x,0) = 1 & u_t(x,0) = 2 \\ u(0,t) = 1 & u(2,t) = 2 \end{cases}$$

Answer Key

1) $\frac{\pi}{2}+\frac{1}{2}\sin\left(\frac{\pi}{2}+\frac{1}{4}\right)-\frac{1}{2}\sin\left(\frac{\pi}{2}+\frac{1}{4}\right)$

2) $\frac{5}{16}$

3) $u\left(\frac{1}{2},\frac{1}{2}\right)>v\left(\frac{1}{2},\frac{1}{2}\right)$

4) $2\frac{1}{192}$

Duhamel's Principle

Questions

1) Solve the following nonhomogeneous Wave Equation.

$$\begin{cases} u_{tt} = u_{xx} + \sin(\pi x) & 0 \le x \le 1, \quad t \ge 0 \\ u(x,0) = u_t(x,0) = 0 \\ u(0,t) = u(1,t) = 0 \end{cases}$$

2) Solve the following nonhomogeneous Wave Equation.

$$\begin{cases} u_{tt} = u_{xx} + \cos(\pi x) & 0 \le x \le 1, \quad t \ge 0 \\ u(x,0) = u_t(x,0) = 0 \\ u_x(0,t) = u_x(1,t) = 0 \end{cases}$$

3) Use d'Alembert's formula for the homogeneous wave equation and Duhamel's principle

to prove the nonhomogeneous case of d'Alembert's formula.

I.e. show that the solution to the problem:

$$\begin{cases} u_{tt} = a^2 u_{xx} + F(x,t) & -\infty < x < \infty \,, \; t \ge 0 \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$$

is:

$$u(x,t) = \frac{f(x+at) + f(x-at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} g(s)ds + \frac{1}{2a}\int_0^t \int_{x-a(t-\tau)}^{x+a(t-\tau)} F(s,\tau)dsd\tau$$

Answer Key

1) $u(x,t)=\frac{1}{\pi^2}\sin(\pi x)[1-\cos(\pi t)]$

2) $u(x,t)=\frac{1}{\pi^2}\cos(\pi x)[1-\cos(\pi t)]$

3) $u(x,t)=\frac{f(x+at)+f(x-at)}{2}+\frac{1}{2a}\int_{x-at}^{x+at} g(s)ds+\frac{1}{2a}\int_{0}^{t}\int_{x-a(t-\tau)}^{x+a(t-\tau)} F(s,\tau)dsd\tau$