

Workbook



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Moment Generating Function

Moment Generating Function

Questions

- 1) The following probability function is given for a discrete random variable:
- Find the moment generating function.
 - Derive the expectation from the moment generating function.

x	1	2	3
$p(x)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

- 2) Find the moment generating function of the binomial distribution $X \sim B(n, p)$, and find the first and second moments of the function.
- 3) Find the moment generating function of the geometric distribution $X \sim G(p)$ and calculate the expectation of the probability distribution from the moment generating function.
- 4) Find the moment generating function of the Poisson distribution $X \sim P(\lambda)$. Find the first and second moments of the probability distribution
- 5) Let X be a random variable with the following density function:
 $f(X) = Ae^{-x}$ if $X > 0$ or $X = 0$ and $X < 7$ or $X = 7$, and 0 otherwise.
- Find the value of A .
 - Find the moment generating function of X .
- 6) Let X be a random variable with an expectation of 5 and a variance of 16. Let $M_x(t)$ be the moment generating function of X .
 Y is a random variable with a moment generating function $M_y(t)$.
 Assume $M_y(t) = t \cdot M_x(t)$.
 Calculate the expectation and variance of Y .

Answer Key

- 1) a. $\frac{1}{2}e^t + \frac{1}{3}e^{2t} + \frac{1}{6}e^{3t}$ b. $E(x) = 1\frac{2}{3}$
- 2) $M_x(t) := (e^t \cdot p + 1 - p)^n$, $E(x) = M'_x(t) = n \cdot p$, $E(x^2) = M''_x(t) = np[(n-1)p + 1]$
- 3) $M_x(t) = \frac{e^t p}{1 - e^t \cdot (1 - p)}$, $E(x) = M'_x(t) = \frac{1}{p}$
- 4) $M_x(t) = e^{\lambda(e^t - 1)}$, $E(x) = M'_x(t) = \lambda$, $E(x^2) = M''_x(t) = \lambda(\lambda + 1)$
- 5) a. $A = \frac{1}{1 - e^{-7}}$ b. $M_x(t) = \left(\frac{e^7}{e^7 - 1} \right) \frac{(e^{7(t-1)})}{(t-1)}$
- 6) $E(Y) = 1$, $V(Y) = 9$

Appendix

Probability Distribution	Density Function $f_x(t)$ $f_x(t)$	Cumulative Distribution Function $F_x(t)$	$E(x)$	$Var(x)$	$M_x(t)$
Uniform $U(a,b)$	$f_x(t) = \frac{1}{b-a}$ if $a < t < b$ and $t < b$ or $t = b$ 0 otherwise	$F_x(t) = \begin{cases} 0 & t < a \\ \frac{t-a}{b-a} & a \leq t \leq b \\ 1 & t > b \end{cases}$	$\frac{b+a}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{tb} - e^{ta}}{t(b-a)}$
Exponential $\exp(\lambda)$	$f_x(t) = \lambda e^{-\lambda t}$ if $t > 0$ or $t = 0$ 0 otherwise	$F_x(t) = 1 - e^{-\lambda t}$ if $t > 0$ or $t = 0$ 0 otherwise	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{\lambda}{\lambda - t}$
Normal $N(\mu, \sigma^2)$	$f_x(t) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(t-\mu)^2}{2\sigma^2}}$	$\phi\left(\frac{t-\mu}{\sigma}\right)$	μ	σ^2	$e^{\mu t + \frac{\sigma^2 t^2}{2}}$

Random Variable	Significance	$P_X(x)$	$E(x)$	$Var(x)$	$M_X(t)$
Binomial $Bin(n, p)$	The same Bernoulli trial is independently repeated n times. p is the probability of success, $1-p=q$ is the probability of failure x is the number of successes	$\binom{n}{x} p^x q^{n-x}$ $x = 0, 1, \dots, n$	$n \cdot p$	$n \cdot p \cdot q$	$[pe^t + q]^n$
Geometric $G(p)$	The same Bernoulli trial is independently repeated until the first success. x is the number of trials until the first success.	pq^{x-1} $x = 1, 2, \dots, \infty$	$\frac{1}{p}$	$\frac{q}{p^2}$	$\frac{pe^t}{1-qe^t}$
Poisson $Pois(\lambda)$	x is the number of appearances in a unit of time. The length of time receives values $0, 1, \dots, \infty$	$e^{-\lambda} \frac{\lambda^x}{x!}$	λ	λ	$e^{\lambda(e^t - 1)}$

Characteristics of the Moment Generating Function

Questions

1) Assume that $X_i \sim P(\lambda)$ are independent.

a. Find the moment generating function of $\sum_{i=1}^n X_i$.

b. Prove that $\sum_{i=1}^n X_i \sim P(n \cdot \lambda)$.

2) Assume that $X \sim P(\lambda = 10)$, $Y \sim P(\lambda = 2)$ and X and Y are independent.

a. Find the moment generating function of T .

b. Prove that $T \sim P(\lambda = 12)$.

c. Prove that $X | T = 8 \sim B\left(n = 8, p = \frac{5}{6}\right)$, i.e. that the probability distribution of X ,

given $T = 8$, is binomial with parameters $n = 8$ and $p = \frac{5}{6}$.

3) Assume that $X_i \sim \exp(1)$ $i = 1, 2, \dots, n$ and the variables are independent.

We define $T = \sum_{i=1}^n X_i$.

a. Find the moment generating function of T .

b. Calculate the expectation and variance of T .

c. Let $Z = \frac{T - E(T)}{\sigma(T)}$, i.e. Z is the standardization of T .

Find the moment generating function of Z .

4) The moment generating function of the normal probability distribution is given by the

following formula: $M_X(T) = e^{\mu T + \frac{\sigma^2 T^2}{2}}$ for any T where $X \sim N(\mu, \sigma^2)$.

a. Prove that if $Y = 2X$, then $Y \sim N(2\mu, 4\sigma^2)$.

b. Prove that if $T = X_1 + X_2$ and X_1 and X_2 are independent variables with the same normal probability distribution, then $T \sim N(2\mu, 2\sigma^2)$.

Answer Key

1) a. $M_T(t) = e^{n\lambda(e^t-1)}$

b. Since a. has the same format as $M_x(t) = e^{\lambda(e^t-1)}$, we know that $\sum_{i=1}^n x_i = T \sim P(n\lambda)$.

2) a. $M_T(t) = e^{12(e^t-1)}$ b. $T = X + Y \sim P(\lambda = 12)$

c. $P(x = k | T = 8) = \frac{P(x = k \cap T = 8)}{P(T = 8)}$, we know that $T = X + Y = 8$, so $Y = 8 - X$.

and we can write it as $= \frac{P(X = k \cap Y = 8 - k)}{P(T = 8)} = \frac{P(X = k) \cdot P(Y = 8 - k)}{P(T = 8)}$

$$\frac{\frac{e^{-10} \cdot 10^k}{k!} \cdot \frac{e^{-2} \cdot 2^{8-k}}{(8-k)!}}{\frac{e^{-12} \cdot 12^8}{8!}} = \frac{e^{-10} \cdot 10^k}{k!} \cdot \frac{e^{-2} \cdot 2^{8-k}}{(8-k)!} \cdot \frac{8!}{e^{-12} \cdot 12^8} = \frac{8!}{k!(8-k)!} \cdot \frac{10^k \cdot 2^8}{2^k \cdot 12^8}$$

$$\begin{aligned} \binom{8}{k} 5^k \cdot \left(\frac{1}{6}\right)^8 &= \binom{8}{k} \frac{5^k}{6^k} \left(\frac{1}{6}\right)^8 \cdot 6^k = \binom{8}{k} \binom{5}{6}^k \left(\frac{1}{6}\right)^8 \cdot \frac{1}{6} \\ &= \binom{8}{k} \binom{5}{6}^k \binom{1}{6}^{k-8} \sim B\left(n = 8, p = \frac{5}{6}\right) \end{aligned}$$

3) a. $M_T(t) = \left(\frac{1}{1-t}\right)^n$ b. $E(T) = n$ $E(T^2) = n(n+1)$ $V(T) = n$

c. $M_Z(t) = e^{-n^{\frac{1}{2}}t} \left(\frac{1}{1 - n^{\frac{1}{2}}t}\right)^n$

4) a. Y is a linear transformation, so $Y = aX + b$, and therefore $a = 2$, $b=0$.

Since $M_{aX+b}(t) = e^{bt} \cdot M_X(at)$, we get $M_y(t) = e^{0t} \cdot e^{\mu \cdot 2t + \frac{\sigma^2(2t)^2}{2}} = 1 \cdot e^{2\mu t + \frac{4\sigma^2 t^2}{2}} = e^{(2\mu)t + \frac{(4\sigma^2)t^2}{2}}$
which means $y \sim N(2\mu, 4\sigma^2)$.

b. We know that $T = X_1 + X_2$, and also that $X_1 \sim N(\mu, \sigma^2)$ and the same for X_2 .
Using the formula of moment generating function of two random variables:

$$M_{T=X_1+X_2}(t) = \left(e^{\mu t + \frac{\sigma^2 t^2}{2}}\right)^2 = e^{(2\mu)t + \frac{(2\sigma^2)t^2}{2}}$$

which is the same format of the given normal probability distribution, so $T \sim N(2\mu, 2\sigma^2)$.