proprep

Workbook

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First Order Equations

Method of Cliaracteristics

Questions

1) Solve the following equation with initial condition. $\alpha \neq \frac{1}{2}$ is a constant.

$$2u_x + u_y = 0 \qquad \gamma = \{(x,y) \mid y = \alpha x\} \qquad u|_\gamma = x^2 + y^2$$

2) Solve the following equation with initial condition.

$$3u_x - 2u_y = 0\ , \quad \gamma = \{(x,y) \mid y = x^2,\ x \geq 0\}\ , \quad u|_\gamma = x - y$$

3) Solve the following equation with initial condition.

$$u_x + 2u_y = 0\ , \quad \gamma = \{(x,y) \mid y = x^2,\ x \leq 0\}\ , \quad u|_\gamma = x + \sin(xy)$$

4) Solve the following equation with initial condition.

$$u_x - u_y = -u\ , \quad \gamma = \{(x,y) \mid y = 0\}\ , \quad u|_\gamma = x^2 - x^4$$

5) Solve the following equation with initial condition.

$$2u_x - 3u_y + 2u = 0, \qquad u(x,-x) = (x+1)e^{-x}$$

6) Solve the following equation with initial condition.

$$\begin{cases} u_x + u_y + u = (2x+1)e^{x^2} \quad (y \geq e^{-x}) \\ u(x, e^{-x}) = e^{x^2} + e^{-x} \end{cases}$$

7) Given that $u(x,y)$ is a solution of the problem $y^2 u_x + u_y = -u$ in the 1st quadrant $[x \geq 0,\ y \geq 0]$

and satisfies the initial conditions $u(x,0) = 1\ (x > 0)$ and $u(0,y) = 0\ (y > 0)$.

Compute $u\left(\frac{1}{5}, 1\right)$.

8) Solve the following problem, where a is a real constant.

$$\begin{cases} 2u_x + u_y = -u & 0 \leq y \leq x \\ u(x,0) = a\cos x + \sin y & x > 0 \\ u(y,y) = 0 & y > 0 \end{cases}$$

Answer Key

1) $u(x,y)=(1+\alpha^2)\left(\dfrac{x-2y}{1-2\alpha}\right)^2$

2) $u(x,y)=\dfrac{1}{3}+\sqrt{\dfrac{1}{9}+\dfrac{1}{3}(2x+3y)}-\left(\dfrac{1}{3}+\sqrt{\dfrac{1}{9}+\dfrac{1}{3}(2x+3y)}\right)^2$

3) $u(x,y)=\left(1-\sqrt{1+y-2x}\right)+\sin\left(1-\sqrt{1+y-2x}\right)^3$

4) $u(x,y)=\left((x+y)^2-(x+y)^4\right)e^y$

5) $u(x,y)=(3x+2y+1)e^{-3x-2y}e^{2x+2y}$

6) $u(x,y)=e^{x^2}+e^{-x}$

7) $u\left(x=\dfrac{1}{5},y=1\right)=0$

8) $u(x,y)=\begin{cases} ae^{-y}\cos(x-2y)+e^{-y}\sin(x-2y) & x>2y\geq 0 \\ 0 & 2y>x\geq y\geq 0 \end{cases}$

Lagrange's Method

Questions

1) Given the PDE $xu_x + yuu_y = u \quad (x, y, u > 0)$.

Find the general solution using Lagrange's Method.

2) Given the PDE $x^2u_x + y^2u_y = u^2$.

Find the general solution using Lagrange's Method.

3) Given the PDE $xu \cdot u_x + yu \cdot u_y = -xy \quad (x, y, u > 0)$.

Find the general solution using Lagrange's Method.

4) Given the PDE $(y^2 + u^2)u_x - xyu_y = -xu$.

Find the general solution using Lagrange's Method.

5) Given the PDE $xu_x + yu_y = 2xy \quad (x, y > 0)$.

a) Find the general solution using Lagrange's Method.

b) Find the solution which satisfies $u(x,1) = x \quad (x > 0)$.

6) Solve the PDE $e^y u_x - e^x u_y = -e^{x+y}u \quad (u > 0)$ with the condition $u(x,0) = 1$.

7) Given the PDE $\frac{1}{e^x\sqrt{y+1}}u_x + yu_y = y^2u \quad (u, y > 0)$.

a) Find the general solution using Lagrange's Method.

b) Verify that the solution does indeed satisfy the PDE.

8) Given the PDE $(\cos y)u_x + (\sin y)u_y = e^y(\sin y)u \quad (u > 0,\ 0 < y < \frac{\pi}{2})$.

Find the general solution using Lagrange's Method.

9) Given the PDE $\frac{1}{y}u_x + \frac{1}{x}u_y = 2 \quad (x, y > 0)$.

Find the general solution using Lagrange's Method.

10) Given the PDE $(x+y)u_x + (y-x)u_y = x^2 - y^2 \quad (x, y > 0)$.

Find the general solution using Lagrange's Method.

11) Given that $u(x,y) = \frac{e^y}{y+1}F\left(\frac{y+1}{x^2+1}\right)$ is the general solution to a PDE of the form $A(x,y,u)u_x + B(x,y,u)u_y = C(x,y,u)$.

a) Find the functions A, B, C.

b) Find a particular solution to the PDE which satisfies $u(0,y) = y^2$.

Answer Key

1) $F\left(\frac{x}{u}, \ln y - u\right) = 0$

2) $u(x,y) = \dfrac{1}{F\left(\frac{1}{y} - \frac{1}{x}\right) + \frac{1}{x}}$

3) $u = \sqrt{F\left(\frac{x}{y}\right) - xy}$

4) $F\left(\frac{y}{u}, x^2 + y^2 + u^2\right) = 0$

5) a) $u(x,y) = xy - F\left(\frac{x}{y}\right)$

b) $u(x,y) = xy$

6) $u(x,y) = e^{e^y - 1}$

7) a) $u(x,y) = e^{\frac{1}{2}y^2 - F\left(\frac{1}{\sqrt{y}}e^{\frac{1}{2}x} + \frac{1}{3}e^{\frac{3}{2}x}\right)}$

b) Verify

8) $u = e^{e^y - F(e^{-x}\sin y)}$

9) $u(x,y) = xy - F\left(\frac{x}{y}\right)$

10) $u(x,y) = \dfrac{x^2 - 2xy - y^2 - F\left(\ln\left(\sqrt{x^2 + y^2}\right) + \arctan\left(\frac{y}{x}\right)\right)}{4}$

11) a)

$$A(x,y,u)=x^2+1$$

$$B(x,y,u)=2x(y+1)$$

$$C(x,y,u)=2xyu$$

b)

$$u(x,y)=\frac{e^y}{y+1}\frac{\left(\frac{y+1}{x^2+1}-1\right)^2}{e^{\frac{y+1}{x^2+1}-1}}\frac{y+1}{x^2+1}$$